

Question 8

For $u_t = u_{xx}$ on $0 < x < L$, $t \geq 0$, with $L > 0$, impose

$$u(0, t) = 0, \quad u_x(L, t) + hu(L, t) = 0, \quad h \in \mathbb{R}.$$

A separated mode obeys $X'' + \lambda X = 0$, $T' = -\lambda T$. Define the dimensionless parameter $q = hL$. No sign restriction is imposed on h ; a negative value need not model passive heat loss.

Tasks

1. Derive the identity $\lambda \int_0^L X^2 = \int_0^L (X')^2 + hX(L)^2$. Show that $h \geq 0$ guarantees positive eigenvalues, and determine exactly when $\lambda = 0$ has a nonzero factor.
2. For $\lambda > 0$, derive an eigenvalue equation using $z = L\sqrt{\lambda}$. For $\lambda < 0$, use $\rho = L\sqrt{-\lambda}$ to derive $\rho \coth \rho = -q$.
3. Prove that the negative-eigenvalue equation has exactly one root if $q < -1$ and none if $q \geq -1$. Determine the time behavior of this mode and of the zero mode.
4. For $q = -2$, bracket the positive root ρ to within 0.001, state the corresponding eigenvalue in terms of L , and sketch the intersection defining it. Mark the limiting value at $\rho = 0$ as a limit, not a negative eigenvalue.

Question 8 – Solution

Strategy. A Robin boundary term can change the sign of the eigenvalue; analyze zero before treating the oscillatory and hyperbolic cases.

Step 1: Retain the boundary contribution. Integration by parts gives $\lambda \int X^2 = \int (X')^2 - [XX']_0^L = \int (X')^2 + hX(L)^2$. For $h \geq 0$, equality at zero would force $X' = 0$ and then $X = 0$; nonzero factors therefore have $\lambda > 0$. At $\lambda = 0$, $X = Ax$ and the right condition is $A(1 + hL) = 0$. Thus a nonzero zero mode exists exactly when $\boxed{q = -1}$.

Step 2: Separate the positive and negative cases. For $\lambda > 0$, $X = A \sin(\sqrt{\lambda}x)$ and

$$\boxed{z \cos z + q \sin z = 0, \quad z > 0.}$$

For $\lambda = -\mu^2 < 0$, $X = A \sinh(\mu x)$ and the right condition becomes $\mu \cosh(\mu L) + h \sinh(\mu L) = 0$. Since $\rho = \mu L > 0$,

$$\boxed{\rho \coth \rho = -q, \quad \lambda = -\rho^2/L^2.}$$

Dividing by $\sinh \rho$ here is valid only in this strictly negative case.

Step 3: Prove the threshold and uniqueness. Let $F(\rho) = \rho \coth \rho$. It has limit 1 as $\rho \downarrow 0$ and $F(\rho) \rightarrow \infty$. Its derivative is $F' = (\sinh \rho \cosh \rho - \rho) / \sinh^2 \rho > 0$: the numerator is zero at zero and has derivative $2 \sinh^2 \rho > 0$ for $\rho > 0$. Therefore its range is $(1, \infty)$, giving exactly one negative eigenvalue iff $q < -1$. That mode grows as $e^{\rho^2 t/L^2}$. At $q = -1$, the zero mode $X = x$ is stationary; it is not a negative eigenmode.

Step 4: Quantify the concrete crossing. For $q = -2$, numerical substitution gives $F(1.915) < 2 < F(1.916)$. Strict monotonicity certifies one root in this interval, approximately $\rho = 1.915008$. Thus $\lambda \approx -3.667256/L^2$. The open point at $(0, 1)$ records the excluded endpoint limit of F .

