

**Question 7**

On  $1 < x < e$ ,  $t \geq 0$ , consider the variable-coefficient equation

$$\frac{1}{x}u_t = (xu_x)_x, \quad u(1, t) = u(e, t) = 0.$$

For a nonzero product  $u = X(x)T(t)$ , use the separation convention  $(xX')' + \lambda X/x = 0$ ,  $T' = -\lambda T$ . Introduce  $s = \log x$  and  $Z(s) = X(e^s)$ .

**Tasks**

1. Derive the factor equations from the PDE and verify, by the chain rule, the transformed spatial equation and its endpoint conditions in  $s$ .
2. Determine every eigenvalue and spatial factor and verify the resulting separated fields in the original  $x$  equation.
3. Derive an integral identity that proves eigenvalue positivity. Identify the weight in the squared spatial norm; explain why an unweighted denominator would not follow from this equation.
4. Prove weighted orthogonality of distinct spatial modes by subtracting their ODE identities. Evaluate their squared weighted norms and locate the interior zeros of the third mode in the original coordinate.

## Question 7 – Solution

**Strategy.** A change of spatial coordinate can simplify separation, but the original equation determines the correct integration weight.

**Step 1: Transform the spatial equation carefully.** Substitution gives  $XT'/x = (xX')'T$ . Nonzero anchors yield  $(xX')' = -\lambda X/x$  and  $T' = -\lambda T$ . Since  $X' = Z_s/x$ , one has  $xX' = Z_s$  and  $(xX')' = Z_{ss}/x$ . Multiplying the spatial equation by  $x$  gives

$$Z_{ss} + \lambda Z = 0, \quad Z(0) = Z(1) = 0.$$

The physical endpoints  $1, e$  become  $0, 1$ , not an interval of length  $e - 1$ .

**Step 2: Recover the modes in the original variable.** The transformed Dirichlet problem has  $\lambda_n = n^2\pi^2$  and  $Z_n(s) = \sin(n\pi s)$  for  $n \geq 1$ . Thus

$$u_n(x, t) = C \sin(n\pi \log x) e^{-n^2\pi^2 t}.$$

Indeed  $xX'_n = n\pi \cos(n\pi \log x)$  and  $(xX'_n)' = -n^2\pi^2 X_n/x$ . Multiplication by the time factor gives exactly  $u_t/x$ . At  $x = 1, e$  the sine values vanish. Zero and negative eigenvalues produce only the trivial Dirichlet factor.

**Step 3: Derive the weighted energy quotient.** Multiplication of  $-(xX')' = \lambda X/x$  by  $X$  and integration give

$$\lambda \int_1^e \frac{X^2}{x} dx = \int_1^e x(X')^2 dx.$$

The endpoint term vanishes. For a nonzero factor the denominator is positive; zero numerator would force a constant and then the zero field. Hence  $\lambda > 0$ . The weight  $1/x$  comes from the coefficient of  $u_t$ ; dropping it would change the identity and the associated eigenvalue quotient.

**Step 4: Verify orthogonality, norms and nodes.** For distinct modes  $m, n$ , subtracting the two integrated spatial identities cancels the derivative terms and yields  $(\lambda_n - \lambda_m) \int_1^e X_n X_m dx/x = 0$ . Thus their weighted inner product vanishes. With  $s = \log x$ ,

$$\int_1^e \frac{X_n^2}{x} dx = \int_0^1 \sin^2(n\pi s) ds = \frac{1}{2}.$$

For  $n = 3$ , the interior zeros occur at  $s = 1/3, 2/3$ , hence at  $x = e^{1/3}, e^{2/3}$ . Equal spacing in  $s$  is not equal spacing in  $x$ .