

Question 4

A product field $u(x, t) = X(x)T(t)$ has proportional spatial snapshots. Consider the two-mode heat solution on $0 \leq x \leq \pi$, $t \geq 0$,

$$u(x, t) = e^{-t} \sin x + e^{-4t} \sin(2x).$$

For any two positions and times define

$$D = u(x_1, t_1)u(x_2, t_2) - u(x_1, t_2)u(x_2, t_1).$$

Tasks

1. Prove that a product field has $D = 0$. Conversely, if every such determinant vanishes and $u(x_0, t_0) \neq 0$ somewhere, construct factors X, T representing u everywhere.
2. Verify the PDE $u_t = u_{xx}$ and homogeneous Dirichlet endpoints for the displayed sum. Compute D at $x_1 = \pi/2, x_2 = \pi/6, t_1 = 0, t_2 = \log 2$ and decide whether the sum is a single product.
3. Find every interior spatial zero of $u(\cdot, t)$, the time at which it leaves the interior, and the sign of the field after that time.
4. Sketch the normalized snapshots $v(x, t) = e^t u(x, t)$ at $t = 0$, $(\log 2)/6$ and $(\log 2)/3$. Explain why division by a nonzero time factor cannot make a nonseparable field separable.

Question 4 – Solution

Strategy. Test separation with an invariant determinant, then interpret the changing spatial shape.

Step 1: Establish an exact product test. For a product, both terms of D equal $X(x_1)X(x_2)T(t_1)T(t_2)$. Conversely, use the nonzero anchor to write $u(x, t)u(x_0, t_0) = u(x, t_0)u(x_0, t)$. Hence

$$\boxed{X(x) = u(x, t_0), \quad T(t) = u(x_0, t)/u(x_0, t_0)}$$

gives a product everywhere, including zeros. The nonzero-anchor restriction excludes only the identically zero field, which is already a trivial product.

Step 2: Verify the solution and reject one-product form. Each term has time derivative equal to its second spatial derivative; the sum solves the PDE and vanishes at $x = 0, \pi$. At the stated points,

$$D = \frac{\sqrt{3}}{2} \left(\frac{1}{16} - \frac{1}{2} \right) = \boxed{-\frac{7\sqrt{3}}{32} \neq 0}.$$

Thus superposition preserves the linear PDE but does not preserve the class of single products. The two spatial modes decay at different rates.

Step 3: Track the moving interior node. Factoring gives $u = e^{-t} \sin x [1 + 2e^{-3t} \cos x]$. On $0 < x < \pi$, $\sin x > 0$, so the only possible interior zero is

$$x_*(t) = \arccos(-e^{3t}/2), \quad 0 \leq t < t_* := \frac{\log 2}{3}.$$

It moves from $2\pi/3$ toward π . At $t = t_*$ it is at the endpoint, not an interior zero. For $t \geq t_*$ the field is strictly positive in the interior. Before then it is positive to the left of the node and negative to its right.

Step 4: Interpret the normalized plots. The three normalized fields are $\sin x + \sin 2x$, $\sin x + 2^{-1/2} \sin 2x$, and $\sin x + \frac{1}{2} \sin 2x$. Their changing zeros reveal changing shape. If $e^t u$ were a product, multiplying its time factor by e^{-t} would make u a product too, contradicting the determinant. Normalization changes amplitudes but cannot repair this failure of separation.

