

Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, take $\kappa, L > 0$ and insulated endpoints $u_x(0, t) = u_x(L, t) = 0$. Use $X'' + \lambda X = 0$ and $T' = -\kappa \lambda T$. All fields are smooth enough for the stated derivatives and integrals.

Tasks

1. Solve the spatial eigenvalue problem, including $\lambda = 0$, and exclude $\lambda < 0$ by an integral identity.
2. Show directly from the PDE that the spatial mean is constant in time. Prove that each positive-eigenvalue spatial factor has zero mean, without evaluating its trigonometric integral.
3. Construct a finite separated-mode sum satisfying $u(x, 0) = 2 + 3 \cos(\pi x/L)$. Verify the PDE, both endpoints and the initial data, and identify the limiting field as $t \rightarrow \infty$.
4. Explain precisely what is lost if one divides the eigenvalue calculation by λ and discards the zero case. Can any finite sum of positive-eigenvalue modes reproduce these initial data?

Question 3 – Solution

Strategy. The zero eigenvalue carries a conserved quantity and must be analyzed before any division by the separation constant.

Step 1: Retain the constant spatial mode. Integration by parts gives $\lambda \int X^2 = \int (X')^2 \geq 0$ because both endpoint derivatives vanish. Thus negative eigenvalues are excluded. For $\lambda = 0$, $X = ax + b$ and the endpoints force $a = 0$; nonzero constants are allowed. For $\lambda = k^2 > 0$, the left derivative condition removes the sine term and the right requires $\sin(kL) = 0$. Consequently

$$\lambda_0 = 0, X_0 = 1; \quad \lambda_n = (n\pi/L)^2, X_n = \cos(n\pi x/L), n \geq 1.$$

Step 2: Track the conserved mean. For $m(t) = L^{-1} \int_0^L u(x, t) dx$, $m'(t) = \kappa[u_x]_0^L/L = 0$. Integrating $X'' + \lambda X = 0$ gives $\lambda \int_0^L X dx = -[X']_0^L = 0$. Thus every $\lambda > 0$ factor has zero mean. The zero mode has a constant time factor and can carry a nonzero mean.

Step 3: Verify the two-mode construction. Let $\omega = \kappa(\pi/L)^2$. Then

$$u(x, t) = 2 + 3e^{-\omega t} \cos(\pi x/L).$$

Its time derivative is $-3\omega e^{-\omega t} \cos(\pi x/L)$, equal to κu_{xx} . Its endpoint derivatives vanish, and its initial trace is the prescribed function. Its mean is two for every time. Since the decaying term has absolute value at most $3e^{-\omega t}$, it tends uniformly to $u = 2$. No infinite-series argument is required for this finite construction.

Step 4: Diagnose the missing mode. Dividing by λ loses the legitimate constant eigenfunction and the corresponding stationary PDE solutions. Every finite linear combination of positive-eigenvalue modes has zero spatial mean at each time, while the specified initial data have mean two. Such a combination cannot reproduce these data. This obstruction follows from the integral, not from an inability to find enough trigonometric coefficients.