

Question 2

Seek nonzero separated solutions of $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and mixed endpoint conditions

$$u(0, t) = 0, \quad u_x(L, t) = 0.$$

Use the convention $X'' + \lambda X = 0$, $T' = -\kappa \lambda T$.

Tasks

1. Derive the endpoint conditions on X . Use integration by parts to prove that every nonzero spatial factor has $\lambda > 0$; address equality explicitly.
2. Solve the spatial problem and list all eigenvalues and eigenfunctions, using an index $n = 0, 1, 2, \dots$. Explain why checking only a sine condition at both ends would give the wrong spectrum.
3. Find the time factor and compare the amplitude half-lives of the first two modes. Distinguish amplitude half-life from the half-life of squared amplitude.
4. Sketch the first three spatial factors normalized to have sine coefficient one, using $\xi = x/L$. Locate their interior zeros and verify their right-end slopes.

Question 2 – Solution

Strategy. Let the actual boundary operators select the admissible separation constants.

Step 1: Prove positivity before solving the ODE. At a time where $T \neq 0$, the endpoint conditions give $X(0) = 0$, $X'(L) = 0$. Multiplication by X and integration yield

$$\lambda \int_0^L X^2 dx = \int_0^L (X')^2 dx - [XX']_0^L = \int_0^L (X')^2 dx.$$

If $\lambda = 0$, then X' vanishes identically; $X(0) = 0$ forces $X = 0$. Thus a nonzero factor has $\boxed{\lambda > 0}$; negative values are impossible.

Step 2: Apply the two different endpoint conditions. Put $k = \sqrt{\lambda}$. The general solution is $A \cos(kx) + B \sin(kx)$. The left condition gives $A = 0$, and the right gives $Bk \cos(kL) = 0$. For $B \neq 0$,

$$\boxed{k_n = \frac{(n+1/2)\pi}{L}, \quad \lambda_n = k_n^2, \quad X_n(x) = \sin(k_n x), \quad n = 0, 1, \dots}$$

The right condition is on the derivative, so it selects zeros of cosine. Using $\sin(kL) = 0$ would impose an unrequested Dirichlet condition there.

Step 3: Compare decay measures. The modes are $CX_n(x)e^{-\kappa\lambda_n t}$. Their amplitude half-lives are $\tau_n = \log 2/(\kappa\lambda_n)$, so $\tau_1 = \tau_0/9$. Squared amplitude decays as $e^{-2\kappa\lambda_n t}$ and has half-life $\tau_n/2$. Confusing those two quantities introduces a factor of two.

Step 4: Check the plotted nodes and slopes. For $n = 0, 1, 2$, the normalized factors are $\sin((n+1/2)\pi\xi)$. There are respectively zero, one and two interior zeros: none; $2/3$; and $2/5, 4/5$. Every right slope is zero because $\cos((n+1/2)\pi) = 0$. The right values alternate $1, -1, 1$ and need not vanish.

