

Question 1

Let I, J be open intervals, $\kappa > 0$, $r \in \mathbb{R}$, and consider

$$u_t = \kappa u_{xx} + ru \quad ((x, t) \in I \times J).$$

A separated field is $u(x, t) = X(x)T(t)$, with $X \in C^2(I)$ and $T \in C^1(J)$. Assume neither factor is identically zero. Do not assume either factor is nonzero at every point.

Tasks

1. Substitute the product into the PDE. Choose one time where T is nonzero and use it to derive a spatial ODE on all of I , without dividing by X .
2. Show that a single real constant λ gives $X'' + \lambda X = 0$ and $T' = (r - \kappa\lambda)T$ everywhere on their respective intervals. Prove the converse as well.
3. Determine whether a nontrivial T can vanish at a time in J . Explain why zeros of X do not invalidate the separated field and what happens if a factor is identically zero.
4. Verify $X(x) = \sin(2x)$ when $I = \mathbb{R}$. Find all corresponding time factors, determine the growth threshold in r , and explain the harmless freedom to rescale the two factors reciprocally.

Question 1 – Solution

Strategy. Evaluate at a nonzero anchor value of one factor before deriving equations valid even at zeros of the other.

Step 1: Derive a global spatial identity. Substitution yields $XT' = \kappa X''T + rXT$. Choose $t_0 \in J$ with $T(t_0) \neq 0$. Evaluating there gives, for every $x \in I$,

$$\kappa X''(x) = \left(\frac{T'(t_0)}{T(t_0)} - r \right) X(x).$$

Set $\lambda = (r - T'(t_0)/T(t_0))/\kappa$. Then $\boxed{X'' + \lambda X = 0}$ on all of I , including any zeros of X .

Step 2: Recover the time equation and the converse. Substituting $X'' = -\lambda X$ into the product identity gives $X[T' - (r - \kappa\lambda)T] = 0$. Choose x_0 with $X(x_0) \neq 0$ to obtain

$$\boxed{T' = (r - \kappa\lambda)T \quad \text{on } J.}$$

The constant is independent of both variables; a different anchor must give the same value because X is not identically zero. Conversely, these two ODEs imply the original product identity directly, without any divisions. Thus every such pair of factors produces a PDE solution.

Step 3: Treat zeros and the excluded trivial field. The time equation gives $T(t) = Ce^{(r-\kappa\lambda)(t-t_0)}$. For a nontrivial factor $C \neq 0$, it never vanishes. Spatial zeros are allowed; the identities above remain valid there. A factor identically zero produces $u \equiv 0$, which solves the PDE but determines no separation constant from its chosen factors. The trivial field must be retained separately, not mistakenly assigned restrictions obtained by division.

Step 4: Verify a concrete mode and its normalization. For $X = \sin(2x)$, $X'' = -4X$, so $\lambda = 4$ and

$$u(x, t) = C \sin(2x) e^{(r-4\kappa)t}.$$

Its nonzero amplitude grows if $r > 4\kappa$, is constant if $r = 4\kappa$, and decays if $r < 4\kappa$. Direct differentiation verifies the PDE, including at $x = n\pi/2$. Replacing (X, T) by $(aX, T/a)$ for $a \neq 0$ leaves u unchanged; the two factors are not individually unique until a normalization is chosen.