

Question 10

On the unit square $\Omega = (0, 1)^2$, consider a C^2 field smooth up to each edge, satisfying

$$\Delta u = s, \quad \partial_n u = g \text{ on } \partial\Omega,$$

where s is a constant and g is constant on each open edge. Denote these four constants by g_L, g_R, g_B, g_T . At vertices, interpret boundary data by one-sided edge traces. You may use the divergence theorem and Green's identity.

Tasks

1. Identify the boundary type and derive a necessary relation between s and the four edge values, retaining the outward-normal signs.
2. For arbitrary edge constants satisfying that relation, construct a polynomial solution of the form $u = ax^2 + by^2 + cx + dy + C$. Verify that the compatibility condition is also sufficient for this data family.
3. Prove that any two solutions for the same compatible data differ by a constant. Explain why this is uniqueness up to a constant rather than uniqueness of the original problem.
4. Take $(g_L, g_R, g_B, g_T) = (0, 2, 0, 4)$. Find s , impose the normalization $\int_{\Omega} u \, dA = 0$, and give the resulting unique solution. Would the same edge data be compatible with $s = 0$?

Question 10 – Solution

Strategy. Separate existence compatibility from the constant ambiguity left by Neumann data.

Step 1: Integrate the equation before solving it. These are Neumann conditions. Since the square has area one and each edge has length one, the divergence theorem gives

$$s = \int_{\Omega} \Delta u \, dA = \int_{\partial\Omega} \partial_n u \, ds = g_L + g_R + g_B + g_T.$$

In coordinates the derivatives are $-u_x$ on the left, u_x on the right, $-u_y$ on the bottom and u_y on the top. Replacing all by positive coordinate derivatives would give the wrong compatibility relation.

Step 2: Construct every compatible polynomial. For the proposed polynomial the four traces give

$$c = -g_L, \quad 2a + c = g_R, \quad d = -g_B, \quad 2b + d = g_T.$$

Thus

$$u = \frac{g_R + g_L}{2}x^2 + \frac{g_T + g_B}{2}y^2 - g_Lx - g_By + C.$$

Its Laplacian is the sum of the four edge values, exactly s when compatible. Every boundary derivative has the prescribed value. This explicit construction proves sufficiency for constant source and constant edge data on this square; it is not a general sufficiency theorem for arbitrary domains and functions.

Step 3: Prove the exact remaining ambiguity. The difference w of two solutions obeys $\Delta w = 0$, $\partial_n w = 0$. Green's identity yields

$$\int_{\Omega} |\nabla w|^2 \, dA = \int_{\partial\Omega} w \partial_n w \, ds - \int_{\Omega} w \Delta w \, dA = 0.$$

Continuity gives $\nabla w = 0$ everywhere, and connectedness of the square makes w constant. Conversely any added constant leaves all equations and data unchanged. A value or mean normalization is needed for actual uniqueness.

Step 4: Normalize the concrete field. Here $s = 6$ and $u = x^2 + 2y^2 + C$. Its integral is $1/3 + 2/3 + C = 1 + C$. The zero-mean condition fixes $C = -1$, so $u = x^2 + 2y^2 - 1$ is unique among solutions with that normalization. For $s = 0$, the integrated compatibility condition would require $0 = 6$, so no solution exists with these edge data.