

Question 9

A problem is well posed in specified function spaces if solutions exist, are unique, and depend continuously on the prescribed data. To examine the third requirement, consider the Cauchy data problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y \leq 1; \quad u(x, 0) = f(x), \quad u_y(x, 0) = g(x),$$

with $u(0, y) = u(\pi, y) = 0$. Measure the bottom data by $\|f\|_\infty + \|g\|_\infty$ on $[0, \pi]$ and the observed solution by $\|u(\cdot, 1)\|_\infty$. For integers $n \geq 1$, set

$$u_n(x, y) = \frac{e^{-n}}{n} \sin(nx) \sinh(ny).$$

Tasks

1. Verify the PDE and side conditions and find the exact bottom data f_n, g_n .
2. Compute the bottom data norm and top solution norm. Determine their limits as $n \rightarrow \infty$.
3. Prove that the amplification ratio is unbounded and explain why the calculation rules out a continuous solution map at zero in the stated norms if the problem is uniquely solvable on a linear class containing these fields.
4. Explain why convergence of $u_n(\cdot, 1)$ to zero does not rescue continuity. Rescale the fields to obtain data converging to zero while the top norm stays exactly one, and sketch the vertical amplitude profiles for three n .

Question 9 – Solution

Strategy. Small absolute outputs are not enough for stability; compare their size with the much smaller input, then normalize.

Step 1: Verify the supplied solutions. Two x derivatives multiply u_n by $-n^2$; two y derivatives multiply it by n^2 . Their sum is zero, and $\sin(nx)$ vanishes on both sides. At the bottom, $\boxed{f_n = 0, \quad g_n = e^{-n} \sin(nx)}$. These are smooth fields with compatible side and bottom values.

Step 2: Measure the input and output. For each integer n , $\sup_{[0,\pi]} |\sin(nx)| = 1$. Hence the bottom norm is e^{-n} and the top norm is

$$M_n = \frac{e^{-n}}{n} \sinh n = \frac{1 - e^{-2n}}{2n}.$$

Both tend to zero. Thus this sequence by itself does not contradict continuity; the relative amplification is the information needed for the next step.

Step 3: Expose unbounded amplification. The ratio is $M_n/e^{-n} = \sinh n/n \rightarrow \infty$. If uniqueness holds in a linear solution class containing these fields, solving the homogeneous linear PDE with data is a linear map. A continuous linear map between the stated normed spaces must obey a uniform bound $\|u(\cdot, 1)\|_\infty \leq C(\|f\|_\infty + \|g\|_\infty)$ on its domain. The unbounded ratios rule out that bound. No existence or uniqueness theorem for arbitrary Cauchy data has been assumed.

Step 4: Give a direct discontinuity sequence. Set $w_n = u_n/M_n = \sin(nx) \sinh(ny)/\sinh n$. Its top norm is one, whereas its bottom norm is $n/\sinh n \rightarrow 0$. The zero data have the zero solution; uniqueness would force the solution map to select these w_n , contradicting continuity at zero. The vertical envelopes are $\sinh(ny)/\sinh n$, not solution traces at one fixed x for every n .

