

Question 8

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and

$$u(0, t) = A(t), \quad u(L, t) = B(t), \quad u(x, 0) = f(x).$$

Assume all data are smooth and $f(0) = A(0)$, $f(L) = B(0)$. Define a boundary lifting

$$\ell(x, t) = \left(1 - \frac{x}{L}\right) A(t) + \frac{x}{L} B(t), \quad v = u - \ell.$$

Tasks

1. Determine the PDE, endpoint data and initial data for v . Classify the homogeneity of its PDE and its spatial boundary conditions separately.
2. Find the exact condition on A, B for this affine lifting to leave the transformed PDE homogeneous for every x, t .
3. For $A(t) = t$, $B(t) = 2t$, $f = 0$, compute the transformed source and initial data. Does making the endpoint conditions homogeneous remove the time-dependent forcing from the full problem?
4. Show that a lifting is not unique. Replace ℓ by $\tilde{\ell} = \ell + q$, where $q(0, t) = q(L, t) = 0$, and derive the new source and initial data. Explain why this is a change of unknown, not a different physical field u .

Question 8 – Solution

Strategy. Subtract the lifting from every part of the problem, including the differential equation and the initial condition.

Step 1: Compute the transformed full problem. Since $\ell_{xx} = 0$,

$$v_t - \kappa v_{xx} = -\ell_t = -\left(1 - \frac{x}{L}\right) A'(t) - \frac{x}{L} B'(t).$$

The endpoints are $v(0, t) = v(L, t) = 0$, and $v(x, 0) = f(x) - \ell(x, 0)$. The endpoint conditions are homogeneous Dirichlet. The transformed PDE is linear, but generally nonhomogeneous. Its initial condition is homogeneous only if $f = \ell(\cdot, 0)$.

Step 2: Determine when the source vanishes. For each t , the displayed affine function of x vanishes throughout $(0, L)$ if and only if both its endpoint limits vanish. Thus $A'(t) = B'(t) = 0$ for all t , or equivalently $\boxed{A, B \text{ are constant in time}}$. Nonzero constant boundary values can therefore produce a homogeneous transformed PDE, although the original boundary data were nonhomogeneous.

Step 3: Inspect time-dependent endpoint values. For $A = t, B = 2t, \ell = t(1 + x/L)$, so

$$v_t - \kappa v_{xx} = -(1 + x/L), \quad v(0, t) = v(L, t) = 0, \quad v(x, 0) = 0.$$

The boundary forcing has moved into the interior equation; it has not been removed from the full problem. In particular, $v = 0$ would fail this PDE. The stated value compatibility holds, but no corner-smooth existence is claimed for these data; higher compatibility would require additional checks.

Step 4: Track the freedom in the lifting. Let $\tilde{v} = u - \tilde{\ell} = v - q$. Direct differentiation gives

$$\tilde{v}_t - \kappa \tilde{v}_{xx} = -\ell_t - q_t + \kappa q_{xx}, \quad \tilde{v}(x, 0) = f(x) - \ell(x, 0) - q(x, 0).$$

The endpoints still vanish. For example, $q = x(L - x)e^{-t}$ is a nonzero smooth choice with zero endpoints. Although the source and transformed initial data change, $u = v + \ell = \tilde{v} + \tilde{\ell}$ is identical. Each lifting gives an equivalent formulation when every datum is transformed consistently.