

Question 7

For a real second-order linear PDE in two spatial variables, use the convention

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower-order terms} = 0, \quad D = B^2 - AC.$$

At points where the second-order part is nonzero, call the equation hyperbolic if $D > 0$, elliptic if $D < 0$, and parabolic if $D = 0$. Consider

$$u_{xx} + 2yu_{xy} + xu_{yy} + 3u_x - u = 0.$$

Tasks

1. Identify A, B, C and classify the equation at every point of the plane, including the separating curve. Explain why the coefficient of u_{xy} is not B itself.
2. Classify the points $(0, 1), (2, 1), (1, 1)$, and sketch the type regions with correct axes and boundary curve.
3. At a fixed point, rewrite the principal quadratic form $Q(\xi, \eta) = \xi^2 + 2y\xi\eta + x\eta^2$ as a completed square. Explain its signs and rank in all three regions.
4. Replace $3u_x - u$ by $e^{x+y}u_x + 7u$ and then multiply the entire PDE by a smooth nowhere-zero function $m(x, y)$. Decide which changes can alter the type and justify your answer.

Question 7 – Solution

Strategy. Classify the principal second-order part point by point, keeping the mixed-derivative convention fixed.

Step 1: Read the discriminant correctly. Here $A = 1$, $B = y$ and $C = x$, since the mixed coefficient is $2B = 2y$. Thus $D = y^2 - x$, giving

$x < y^2$: hyperbolic, $x = y^2$: parabolic, $x > y^2$: elliptic.
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Since $A = 1$, the principal part never vanishes completely. The curve is therefore parabolic under the stated convention, not an order-zero exception.

Step 2: Check the points and the geometry. The three discriminants are respectively 1, -1 , 0, so $(0, 1)$ is hyperbolic, $(2, 1)$ elliptic and $(1, 1)$ parabolic. With x horizontal and y vertical, $x = y^2$ is a right-opening parabola; ellipticity lies to its right.

Step 3: Interpret the principal form. Completing the square gives $Q = (\xi + y\eta)^2 + (x - y^2)\eta^2$. For $x > y^2$, it is positive definite, with rank two. For $x < y^2$, the choices $(\xi, \eta) = (1, 0)$ and $(-y, 1)$ give opposite signs; it is indefinite with rank two. On $x = y^2$, it is a nonnegative square of rank one, vanishing on the nonzero direction $(-y, 1)$. This explains the type boundary algebraically.

Step 4: Identify changes that preserve type. Changing lower-order terms leaves A, B, C unchanged. Multiplying by m gives $\bar{D} = (mB)^2 - (mA)(mC) = m^2D$, with the same sign because $m \neq 0$. Neither operation changes type. Allowing $m = 0$ could erase the principal part at those points, which is why the nowhere-zero restriction matters.

