

Question 6

For $u_t + u_x = 0$ on $\mathbb{R}^2$, a classical solution here means a C^1 field satisfying the PDE at every point. Compare

$$u(x, t) = |x - t|, \quad u_\varepsilon(x, t) = \sqrt{(x - t)^2 + \varepsilon^2}, \quad \varepsilon > 0.$$

For this question only, an integral solution is a locally integrable field satisfying

$$\int_{\mathbb{R}^2} u(x, t) (\phi_t + \phi_x)(x, t) \, dx \, dt = 0$$

for every smooth compactly supported test function ϕ .

Tasks

1. Check the PDE for u off $x = t$, and decide whether u is a classical solution on the whole plane.
2. Verify that each u_ε is classical and prove a uniform error bound tending to zero as $\varepsilon \downarrow 0$.
3. Explain precisely why uniform convergence of these classical solutions does not establish that u is classical. Inspect $u_{\varepsilon, x}$ near the line $x = t$.
4. Verify that u is an integral solution using $s = x - t$, $r = t$. State which conclusions about u have and have not been proved.

Question 6 – Solution

Strategy. A solution label includes a regularity requirement; checking a formula away from a singular line is not enough.

Step 1: Locate the failure of classical regularity. For $x > t$, $u_x = 1, u_t = -1$; for $x < t$, $u_x = -1, u_t = 1$. Thus the PDE holds on both open regions. At $x = t$, the two one-sided spatial derivatives disagree. The field is not differentiable there, hence is not a classical solution on $\mathbb{R}^2$.

Step 2: Verify the smooth approximations. Writing $s = x - t$, one has

$$u_{\varepsilon,x} = \frac{s}{\sqrt{s^2 + \varepsilon^2}}, \quad u_{\varepsilon,t} = -\frac{s}{\sqrt{s^2 + \varepsilon^2}}.$$

These are smooth and sum to zero. Also

$$0 \leq \sqrt{s^2 + \varepsilon^2} - |s| = \frac{\varepsilon^2}{\sqrt{s^2 + \varepsilon^2} + |s|} \leq \varepsilon.$$

Equality occurs at $s = 0$, so the uniform error on the whole plane is exactly ε .

Step 3: Separate convergence of values from derivatives. For $s \neq 0$, $u_{\varepsilon,x} \rightarrow \operatorname{sgn}(s)$, while its value at $s = 0$ is always zero. This pointwise limit is discontinuous at zero. It cannot be a uniform limit on any closed interval about zero of these continuous derivatives. Uniform convergence of function values does not preserve differentiability, so it does not prove classical solvability.

Step 4: Check the stated integral formulation. Set $\psi(s, r) = \phi(s + r, r)$. The change of variables has Jacobian one, and $\psi_r = \phi_x + \phi_t$. Compact support and local integrability allow integration in either order. Therefore

$$\int_{\mathbb{R}^2} |x - t|(\phi_t + \phi_x) dx dt = \int_{\mathbb{R}} |s| \left(\int_{\mathbb{R}} \psi_r(s, r) dr \right) ds = 0.$$

The inner integral vanishes by compact support. Thus u is an integral solution under the supplied definition, as well as a classical solution off the line. It remains nonclassical on the whole plane; no uniqueness assertion follows from this verification alone.