

Question 5

For the first-order PDE $u_t + u_x = 0$ on $\mathbb{R}^2$, consider C^1 solutions. A characteristic curve is a curve $(x(t), t)$ along which the PDE forces u to be constant. Suppose data are prescribed on the line $x = at$:

$$u(at, t) = h(t), \quad t \in \mathbb{R},$$

where $a \in \mathbb{R}$ and $h \in C^1(\mathbb{R})$.

Tasks

1. Use the chain rule to identify the characteristic lines, and prove that every C^1 solution has the form $u(x, t) = F(x - t)$.
2. For $a \neq 1$, determine the unique F and verify the resulting full solution for arbitrary h .
3. For $a = 1$, classify exactly which h permit a solution and whether that solution is unique. Give two distinct solutions when $h \equiv H$ is constant.
4. Compare $a = 0, h(t) = \sin t$ with $a = 1, h(t) = \sin t$. Explain geometrically why data on an entire line can determine a unique field, no field, or many fields.

Question 5 – Solution

Strategy. Check whether the data line intersects all characteristic lines or follows just one of them.

Step 1: Recover the characteristic form. Along $x = t + s$, the chain rule gives $\frac{d}{dt}u(t+s, t) = u_x + u_t = 0$. Thus $u(t+s, t) = u(s, 0)$. Defining $F(s) = u(s, 0)$ proves $\boxed{u(x, t) = F(x-t)}$ for every solution. Conversely this form has $u_t = -F'$ and $u_x = F'$, so it solves the PDE.

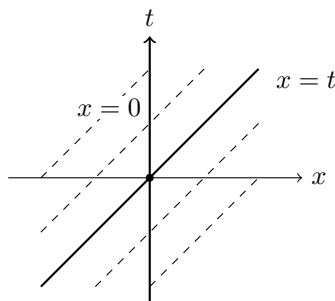
Step 2: Use a noncharacteristic data line. The condition becomes $F((a-1)t) = h(t)$. If $a \neq 1$, the argument ranges over all real numbers exactly once. Consequently

$$\boxed{F(s) = h\left(\frac{s}{a-1}\right), \quad u(x, t) = h\left(\frac{x-t}{a-1}\right)}.$$

This is C^1 , solves the PDE and takes the prescribed trace. The characteristic representation proves uniqueness in the whole stated C^1 class.

Step 3: Diagnose characteristic data. If $a = 1$, the condition reduces to $F(0) = h(t)$ for every t . A solution exists exactly when h is constant, say H . Then any C^1 function with $F(0) = H$ works. For example, $u = H$ and $u = H + x - t$ are distinct solutions. Thus compatible characteristic data leave infinitely many fields undetermined.

Step 4: Interpret the two sine traces. For $a = 0$, $u(x, t) = \sin(t-x)$ is the unique solution. For $a = 1$, the varying trace $\sin t$ contradicts constancy along that characteristic, so no solution exists. The diagram shows the unique intersection with $x = 0$ and the characteristic line $x = t$; the latter supplies information on just one member of the family.



Characteristic family: $x - t = s$