

Question 4

Consider $u_t = u_{xx}$ for $0 < x < 1$, $0 < t \leq T$, with

$$u(x, 0) = f(x) = x(1 - x), \quad u(0, t) = A(t), \quad u(1, t) = B(t).$$

For this question a solution smooth up to the corners has continuous u, u_t, u_x, u_{xx} on the closed rectangle, satisfies the PDE there by continuity, and has $u_t(x, 0)$ and $u_{xx}(x, 0)$ obtained by differentiating its traces. Assume A, B are continuously differentiable.

Tasks

1. Derive the necessary value compatibility conditions at the two initial corners.
2. Derive the next necessary compatibility conditions involving $A'(0), B'(0)$ and f'' .
3. Set $A = B = 0$. Decide whether value compatibility holds and whether a solution in the stated corner-smooth class can exist. Explain why this does not rule out solutions smooth only for positive time.
4. Keep f fixed and construct linear-in-time boundary data for which a polynomial solution exists. Verify the PDE and every datum explicitly; distinguish this sufficiency proof from checking necessary conditions alone.

Question 4 – Solution

Strategy. At a corner, two different traces must describe the same value and the same time derivative.

Step 1: Match the values. Continuity at $(0, 0)$ and $(1, 0)$ requires

$$A(0) = f(0) = 0, \quad B(0) = f(1) = 0.$$

If these equalities fail, even a continuous solution on the entire closed rectangle is impossible. Interior smoothness alone does not imply these equalities.

Step 2: Match the PDE to the time traces. The left boundary trace gives $u_t(0, 0) = A'(0)$ and the initial trace gives $u_{xx}(0, 0) = f''(0)$. The PDE therefore requires $A'(0) = f''(0)$. The right corner gives the analogous identity. Since $f'' = -2$,

$$A'(0) = B'(0) = -2.$$

These conditions use the stated corner regularity; they cannot be imposed silently on a class that has no continuous u_{xx} at the corners.

Step 3: Diagnose homogeneous endpoint data. For $A = B = 0$, both value conditions hold, but $A'(0) = B'(0) = 0 \neq -2$. Thus no solution exists in the corner-smooth class specified here. This is a regularity obstruction at $t = 0$, not a proof of nonexistence in a weaker class. A solution may be continuous up to the initial edge and smooth for $t > 0$ without its second spatial derivative extending continuously through the initial corners. No such weaker solution is needed for the contradiction.

Step 4: Construct a compatible full example. Take $A(t) = B(t) = -2t$ and

$$u(x, t) = x(1 - x) - 2t.$$

Then $u_t = u_{xx} = -2$, $u(x, 0) = f(x)$ and $u(0, t) = u(1, t) = -2t$. This polynomial has all required corner derivatives and proves existence for the modified problem. Matching a finite list of necessary compatibility conditions would not alone prove existence for arbitrary data; the explicit field supplies the sufficiency argument in this case.