

Question 3

Let $u(x, t)$ solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $0 < t < T$, where $\kappa, L, T > 0$. For this problem, initial data are prescribed at $t = 0$; spatial boundary data are prescribed at $x = 0, L$. The outward normal derivative is $\partial_n u = n_x u_x$, with $n_x = -1$ on the left and $n_x = 1$ on the right. Consider

$$u(x, 0) = f(x), \quad u(0, t) = A(t), \quad \kappa \partial_n u(L, t) + hu(L, t) = r(t), \quad h > 0.$$

Dirichlet data prescribe u , Neumann data prescribe $\partial_n u$, and Robin data prescribe a linear combination with nonzero coefficients of both.

Tasks

1. Classify each condition, and identify the portions of the space–time rectangle on which they are prescribed. Is $t = T$ a spatial boundary?
2. Rewrite the right condition with u_x . Write the corresponding coordinate form of $\kappa \partial_n u + hu = r$ if it were imposed at the left endpoint instead.
3. Classify the spatial boundary operator when $h = 0$, and decide exactly when the displayed spatial data are homogeneous. Distinguish this from homogeneity of the PDE and of the initial data.
4. Verify the manufactured field $u_*(x, t) = x^2 + 2\kappa t$. Find f, A, r that make it solve the full displayed problem, and illustrate where each datum acts.

Question 3 – Solution

Strategy. Identify both the operator being prescribed and the geometric location of the data.

Step 1: Locate and name the data. The bottom edge $t = 0$ carries initial displacement/temperature data $f(x)$. The left edge carries Dirichlet data $A(t)$; the right carries Robin data $r(t)$. The top $t = T$ is a final-time slice, not a spatial boundary. No data are prescribed there in this initial-boundary-value problem.

Step 2: Keep the outward signs. At the right, the condition is $\kappa u_x(L, t) + hu(L, t) = r(t)$. At the left, the analogous outward-normal condition would be $-\kappa u_x(0, t) + hu(0, t) = r(t)$. Changing the endpoint changes the sign of the normal derivative, not the sign of the prescribed value by convention.

Step 3: Classify homogeneity at each level. When $h = 0$, the right condition is Neumann, since $\kappa > 0$. The spatial data are homogeneous exactly when $A \equiv 0$ and $r \equiv 0$. The PDE is already linear homogeneous; initial data are homogeneous exactly when $f \equiv 0$. These are three separate assertions. A nonzero Robin coefficient h does not itself make the condition nonhomogeneous.

Step 4: Verify all parts of a full problem. Here $u_{*,t} = 2\kappa$ and $u_{*,xx} = 2$, so the PDE holds. Evaluation gives

$$f(x) = x^2, \quad A(t) = 2\kappa t, \quad r(t) = 2\kappa L + h(L^2 + 2\kappa t).$$

These data also agree at their shared corners. The diagram locates the data; it is a space–time domain, not a graph of the solution value.

