

Question 10

A fixed-end string is driven by

$$\mu u_{tt} = \mathcal{T} u_{xx} + F \sin(kx) \cos(\omega t), \quad k = \pi/L, \quad \omega = k\sqrt{\mathcal{T}/\mu}, \quad F > 0.$$

Both initial displacement and velocity are zero. Test solutions of the supplied form $u(x, t) = q(t) \sin(kx)$; no general separation procedure is required.

Tasks

1. Derive the scalar equation for q and verify $q(t) = Ft \sin(\omega t)/(2\mu\omega)$.
2. Check both initial conditions and both fixed-end conditions. Explain why the factor t does not alter the propagation-speed parameter of the PDE.
3. Compute the energy and verify its rate equals the applied work rate. Evaluate the energy at $t_j = j\pi/\omega$ and describe its growth.
4. Replace the driving frequency by $\Omega > 0$, $\Omega \neq \omega$. Verify the zero-initial-data response $q_\Omega = F(\cos \Omega t - \cos \omega t)/(\mu(\omega^2 - \Omega^2))$ and recover the resonant formula as $\Omega \rightarrow \omega$ for fixed t .

Question 10 – Solution

Strategy. A force at the natural frequency produces growing amplitude while the same local wave equation retains its original speed.

Step 1: Verify the resonant amplitude equation. Substitution gives $q'' + \omega^2 q = (F/\mu) \cos \omega t$. If $D = F/(2\mu\omega)$ and $q = Dt \sin \omega t$, then

$$q' = D(\sin \omega t + \omega t \cos \omega t), \quad q'' + \omega^2 q = 2D\omega \cos \omega t.$$

Thus the proposed response satisfies the forced equation.

Step 2: Check the full initial-boundary data. One has $q(0) = q'(0) = 0$. The spatial sine vanishes at $0, L$, giving both fixed ends for all time. The growing factor is a resonant response to sustained external forcing. It changes the amplitude history, not the coefficient $c^2 = \mathcal{T}/\mu$ that sets the source-free propagation speed.

Step 3: Measure energy and work. Spatial integration gives

$$E(t) = \frac{\mu L}{4}(q'^2 + \omega^2 q^2) = \frac{F^2 L}{16\mu\omega^2} [(\sin \omega t + \omega t \cos \omega t)^2 + (\omega t \sin \omega t)^2].$$

Differentiating the first expression and using the amplitude equation yields $E' = (FL/2)q' \cos \omega t$, exactly $\int_0^L F \sin(kx) \cos(\omega t) u_t dx$. At $t_j = j\pi/\omega$,

$$E(t_j) = \frac{F^2 L}{16\mu} t_j^2.$$

Energy is unbounded along these times. Its instantaneous rate can have either sign; sustained resonant input produces the growing envelope.

Step 4: Take the frequency limit carefully. The supplied q_Ω has zero value and derivative at zero. Direct differentiation gives $q''_\Omega + \omega^2 q_\Omega = (F/\mu) \cos \Omega t$. For fixed t , differentiating numerator and denominator with respect to Ω at ω gives

$$\lim_{\Omega \rightarrow \omega} q_\Omega(t) = \frac{F}{\mu} \frac{-t \sin \omega t}{-2\omega} = \frac{Ft \sin \omega t}{2\mu\omega}.$$

Each fixed nonresonant response is bounded in time, but those bounds are not uniform as its frequency approaches resonance.