

Question 9

Let a sufficiently smooth displacement be periodic in x with period 2π and satisfy $u_{tt} = c^2 u_{xx}$, $c > 0$. Define its mean $m(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} u(x, t) dx$ and momentum $P(t) = \mu \int_{-\pi}^{\pi} u_t dx$, with $\mathcal{T} = \mu c^2$.

Tasks

1. Derive the evolution of the mean and prove momentum conservation.
2. Derive conservation of $E = \frac{1}{2} \int_{-\pi}^{\pi} (\mu u_t^2 + \mathcal{T} u_x^2) dx$. Does bounded energy force bounded displacement for all time?
3. Test the explicit family $u(x, t) = Vt + A \cos x \cos(ct)$. Compute its mean, momentum and energy exactly and use it to answer the preceding question.
4. If the mean initial velocity is zero, prove that a uniform displacement bound does follow from energy and the initial mean. You may use continuity, the intermediate value theorem and Cauchy–Schwarz.

Question 9 – Solution

Strategy. Wave energy controls velocity and slope, but a spatially uniform motion can carry the mean displacement indefinitely.

Step 1: Track the constant spatial mode. Integrating the PDE and using matching periodic slopes gives

$$m''(t) = \frac{c^2}{2\pi} [u_x]_{-\pi}^{\pi} = 0.$$

Hence

$$m(t) = m(0) + m'(0)t, \quad P(t) = 2\pi\mu m'(0).$$

The mean velocity, rather than the mean displacement, is conserved.

Step 2: Derive the periodic energy balance. Multiplication by u_t and integration by parts gives $E' = \mathcal{T}[u_t u_x]_{-\pi}^{\pi} = 0$ by periodicity. The energy contains u_t and u_x , but not u itself. A nonzero constant mean velocity is therefore compatible with a linearly growing mean displacement and a fixed total energy.

Step 3: Verify a drifting example. For $u = Vt + A \cos x \cos(ct)$, the term Vt and the oscillatory term each solve the equation. Orthogonality gives

$$m(t) = Vt, \quad P = 2\pi\mu V, \quad E = \pi\mu V^2 + \frac{\pi\mathcal{T}A^2}{2}.$$

The cosine's kinetic and elastic contributions sum to the second constant. For $V \neq 0$, displacement is unbounded in time even though the energy is finite and constant.

Step 4: Bound displacement when the mean cannot drift. If $m'(0) = 0$, then $m(t) = m(0)$. At each time the continuous function $w = u - m(0)$ has zero spatial mean, so it has a zero at some x_* (or is identically zero). Integrate its derivative from that point over a path of length at most 2π . Cauchy-Schwarz gives

$$|w(x, t)| \leq \sqrt{2\pi} \|u_x(\cdot, t)\|_2 \leq \sqrt{\frac{4\pi E(0)}{\mathcal{T}}}.$$

Thus $|u(x, t)| \leq |m(0)| + \sqrt{4\pi E(0)/\mathcal{T}}$ for all time. The estimate need not be sharp; it identifies the missing mean-velocity condition.