

Question 8

Compare the ideal wave equation with the given dispersive extension

$$u_{tt} = c^2 u_{xx} - \beta u_{xxxx}, \quad c > 0, \quad \beta \geq 0.$$

For a sinusoidal wave $u = \cos(kx - \omega t)$ with $k > 0, \omega > 0$, define phase speed $v_p = \omega/k$ and group speed $v_g = d\omega/dk$. No general theory of wave packets is required.

Tasks

1. Substitute the sinusoid and derive the frequency relation. Determine the units of β .
2. Compute v_p, v_g and compare them for $\beta > 0$. Find their small- k limits and their values when $\beta = 0$.
3. For $\beta > 0$, consider a sum of two nonzero sinusoidal components with distinct positive wave numbers. Prove that the sum cannot translate rigidly with one fixed speed while both components follow the derived positive-frequency relation.
4. Sketch v_p, v_g for $c = 1, \beta = 1/4, 0 < k \leq 3$, showing the limiting value at zero separately. Explain which feature of the ordinary wave equation is lost.

Question 8 – Solution

Strategy. Test sinusoidal components before claiming that an arbitrary composite profile moves without changing shape.

Step 1: Derive the frequency relation. The derivatives give $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$ and $u_{xxxx} = k^4 u$. Therefore

$$\omega^2 = c^2 k^2 + \beta k^4, \quad \omega = k\sqrt{c^2 + \beta k^2}.$$

Since k has units m^{-1} and ω units s^{-1} , β has units m^4/s^2 .

Step 2: Compare the two speeds. Differentiating the positive-frequency branch gives

$$v_p = \sqrt{c^2 + \beta k^2}, \quad v_g = \frac{c^2 + 2\beta k^2}{\sqrt{c^2 + \beta k^2}}.$$

Their difference is $\beta k^2 / \sqrt{c^2 + \beta k^2}$, strictly positive if $\beta > 0$. Both tend to c as $k \downarrow 0$ and both equal c for every k when $\beta = 0$.

Step 3: Test rigid translation of two frequencies. A fixed profile $\cos(k_1 x + \phi_1) + a \cos(k_2 x + \phi_2)$ with $a \neq 0$ translated by speed v gives temporal frequencies vk_1, vk_2 . Independence of the two spatial frequencies requires $v = \omega(k_1)/k_1 = \omega(k_2)/k_2$ for agreement at every x, t . For $\beta > 0$, $v_p(k)$ is strictly increasing for $k > 0$, so distinct wave numbers cannot share this value. Relative phases change in a way that cannot be represented by one common translation.

Step 4: Interpret the comparison. In the ideal equation, all right-moving sinusoidal components travel with the same speed c , so a smooth profile $F(x - ct)$ retains its shape. The given extension makes phase speed depend on frequency; a multi-frequency profile generally changes shape. The graph shows this dependence and the strict separation of the two defined speeds for positive k . The open point at $k = 0$ records their common limit, since the definitions above were made for $k > 0$.

