

**Question 7**

A fixed-end string is known to have displacement

$$u(x, t) = A \sin(\pi x/L) \cos(c\pi t/L), \quad A, L > 0,$$

with unknown speed  $c > 0$ . A snapshot at a known time  $\Delta > 0$  shows  $u(x, \Delta) = 0$  for every  $x$ . Initially the shape and zero velocity are known.

**Tasks**

1. Find every positive speed consistent with the zero snapshot and prove the list is complete.
2. Determine the speed if the independent prior bound  $0 < c < L/\Delta$  is available. Explain why the snapshot alone is insufficient without it.
3. Derive the midpoint velocity at time  $\Delta$  for each admissible speed. Give a consistency test for an exact signed velocity measurement and show that it identifies the speed uniquely when consistent.
4. If the measured velocity is  $-5\pi A/(2\Delta)$ , find  $c$  and predict the midpoint displacement at time  $\Delta/2$ . Verify the signs rather than using only the velocity magnitude.

## Question 7 – Solution

**Strategy.** A snapshot can hide how many oscillations have occurred; a velocity measurement retains both rate and direction.

**Step 1: List all compatible speeds.** Since the spatial sine is not identically zero, the snapshot requires  $\cos(c\pi\Delta/L) = 0$ . For positive  $c$ ,

$$c_n = \frac{L}{\Delta} \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$

Every listed speed gives the same zero snapshot, as well as the specified initial displacement and zero velocity. The positive zeros of cosine prove completeness.

**Step 2: Use an actual prior restriction.** The bound  $c < L/\Delta$  allows only  $n = 0$ , so  $c = L/(2\Delta)$ . Without this independent information, arbitrarily many half-oscillations may have occurred before the snapshot. A zero snapshot does not imply zero speed or determine which crossing of equilibrium was observed.

**Step 3: Recover speed from signed velocity.** At the midpoint,

$$v_n = u_t(L/2, \Delta) = (-1)^{n+1} \frac{\pi A}{\Delta} \left( n + \frac{1}{2} \right).$$

A measured  $v$  is consistent exactly when  $n = |v|\Delta/(\pi A) - 1/2$  is a nonnegative integer and its sign is  $(-1)^{n+1}$ . The strictly increasing magnitudes then select one  $n$  uniquely. In particular, a zero velocity measurement is incompatible with the stated nonzero-amplitude zero snapshot.

**Step 4: Verify the numerical multiple.** For  $v = -5\pi A/(2\Delta)$ , the magnitude gives  $n = 2$  and the predicted sign  $(-1)^3$  is negative, as measured. Thus

$$c = \frac{5L}{2\Delta}, \quad u(L/2, \Delta/2) = A \cos(5\pi/4) = -\frac{A}{\sqrt{2}}.$$

The signed result distinguishes this phase from other times at which the displacement vanishes. Inferring the lowest possible speed without a prior bound would produce the wrong prediction for the intermediate snapshot.