

Question 6

A string on $0 \leq x \leq L$ has fixed ends, constant $\mu, \mathcal{T} > 0$ and equation $\mu u_{tt} = \mathcal{T} u_{xx} + p(x, t)$. Assume sufficient smoothness and define

$$E(t) = \frac{1}{2} \int_0^L (\mu u_t^2 + \mathcal{T} u_x^2) dx.$$

Tasks

1. Derive the energy identity, first retaining boundary power and then using the fixed-end conditions.
2. Prove uniqueness for prescribed initial displacement and velocity when the applied force is prescribed.
3. For $p = 0$ and $A \neq 0$, verify $u = A \sin(\pi x/L) \cos(\omega t)$ with $\omega = (\pi/L)\sqrt{\mathcal{T}/\mu}$. Compute its kinetic, elastic and total energies.
4. Sketch the two energy fractions over $0 \leq \omega t \leq 2\pi$. Explain why a zero-displacement snapshot can have positive energy, and why the energy identity does not claim that displacement at each point is monotone.

Question 6 – Solution

Strategy. Integrate the PDE against velocity so that acceleration and curvature become time derivatives of kinetic and elastic energies.

Step 1: Derive the work balance. Multiplying by u_t and integrating by parts gives

$$E'(t) = \mathcal{T}[u_t u_x]_0^L + \int_0^L p u_t dx.$$

At a fixed endpoint, $u = 0$ for all time, hence $u_t = 0$ there. Thus $E' = \int p u_t$. With no applied force, total energy is constant. The sign of the work rate depends on the force and velocity together.

Step 2: Prove uniqueness for the full initial state. The difference w of two solutions with the same force, endpoints and initial data obeys the homogeneous equation and has zero initial energy. Its energy stays zero, so $w_t = w_x = 0$ everywhere by continuity. The fixed endpoints then force the spatial constant to be zero. Initial displacement alone would not make the kinetic part of the initial difference energy vanish; both initial data are essential.

Step 3: Compute the standing-wave energy. Put $k = \pi/L$, so $\omega^2 = (\mathcal{T}/\mu)k^2$. Then $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$, and both endpoints are zero. Using $\int_0^L \sin^2 kx = \int_0^L \cos^2 kx = L/2$ gives

$$K(t) = E_0 \sin^2(\omega t), \quad V(t) = E_0 \cos^2(\omega t), \quad E_0 = \frac{\mathcal{T} A^2 k^2 L}{4}.$$

Their sum is the constant E_0 .

Step 4: Interpret the exchange of energy. For $A \neq 0$, at $\omega t = \pi/2$ the displacement and elastic energy are zero, but the velocity is nonzero and all energy is kinetic. At $\omega t = 0$ the velocity is zero and all energy is elastic. The field exchanges these forms while oscillating; total-energy conservation does not require pointwise monotonic motion. The graph shows K/E_0 , V/E_0 and their constant sum.

