

Question 5

On $x \geq 0$, use wave speed one and the even profile $F(z) = e^{-z^2}$. For $a > 0$, compare

$$u_D(x, t) = F(x + t - a) - F(x - t + a), \quad u_N(x, t) = F(x + t - a) + F(x - t + a).$$

These formulas include the reflected field from the start; the Gaussian has noncompact tails.

Tasks

1. Verify the wave equation for both fields and show which one satisfies a fixed end and which one satisfies a free end at $x = 0$.
2. Compute their initial displacements and velocities explicitly. Explain why keeping only the incident Gaussian would violate the boundary condition.
3. At $t = a$, compute each field and its velocity. Decide whether zero displacement everywhere implies the string is at rest.
4. For $a = 3$, sketch u_D at $t = 1, 3, 5$ on $0 \leq x \leq 6$. Interpret the sign of the reflected pulse and compare it with the free-end reflection.

Question 5 – Solution

Strategy. An odd or even spatial reflection enforces the corresponding boundary condition.

Step 1: Verify the two boundary models. Each translated profile solves $u_{tt} = u_{xx}$. Since F is even and F' is odd,

$$u_D(0, t) = F(t - a) - F(a - t) = 0, \quad (u_N)_x(0, t) = F'(t - a) + F'(a - t) = 0.$$

Thus u_D satisfies the fixed-end condition and u_N the free-end condition.

Step 2: Recover the actual initial data. Differentiating the time arguments carefully gives

$$\begin{aligned} u_D(x, 0) &= F(x - a) - F(x + a), & (u_D)_t(x, 0) &= F'(x - a) + F'(x + a), \\ u_N(x, 0) &= F(x - a) + F(x + a), & (u_N)_t(x, 0) &= F'(x - a) - F'(x + a), \end{aligned}$$

where $F'(z) = -2ze^{-z^2}$. The incident term alone generally has neither zero boundary displacement nor zero boundary slope. Its reflected partner supplies the exact cancellation; small Gaussian tails are not identically zero.

Step 3: Inspect the collision instant. At $t = a$, the two profile arguments both equal x . Therefore

$$u_D(x, a) = 0, \quad (u_D)_t(x, a) = 2F'(x), \quad u_N(x, a) = 2F(x), \quad (u_N)_t(x, a) = 0.$$

The fixed-end string has zero displacement but nonzero velocity for $x > 0$: it is not at rest. The free-end field has a doubled displacement profile at that instant and zero instantaneous velocity. Displacement and velocity must both be known to specify the state.

Step 4: Interpret the reflected sign. For $a = 3$, the incoming peak at $t = 1$ is near $x = 2$. At $t = 3$ the fixed-end displacement is exactly zero everywhere. At $t = 5$ the reflected pulse is near $x = 2$ with opposite sign. Indeed $u_D(x, 5) = -u_D(x, 1)$. A fixed end reverses displacement upon reflection; a free end preserves its sign. The plotted zero snapshot is an exact cancellation, not disappearance of the motion or its energy.

