

Question 4

Two semi-infinite strings meet at $x = 0$, share tension $\mathcal{T} > 0$ and have densities $\mu_1, \mu_2 > 0$. Set $c_j = \sqrt{\mathcal{T}/\mu_j}$ and $Z_j = \mathcal{T}/c_j$. The massless junction has continuous displacement and continuous transverse force $\mathcal{T}u_x$. For a smooth incident pulse with $0 < \int_{\mathbb{R}} (F')^2 < \infty$, write

$$u_1 = F(t - x/c_1) + RF(t + x/c_1) \quad (x < 0), \quad u_2 = DF(t - x/c_2) \quad (x > 0).$$

Tasks

1. Verify each traveling term solves its local wave equation and derive the two junction equations for R, D .
2. Solve for R, D and classify the sign of the reflected displacement when the second string is denser or lighter.
3. Using rightward power $\mathcal{P} = -\mathcal{T}u_t u_x$, derive the reflected and transmitted energy fractions and prove their sum is one.
4. Evaluate the case $\mu_2 = 9\mu_1$. Analyze the limits $Z_2/Z_1 \rightarrow 0$ and $\rightarrow \infty$, and explain why a transmitted displacement amplitude above one need not violate energy conservation.

Question 4 – Solution

Strategy. Displacement amplitudes and energy fractions differ because the strings have different wave impedances.

Step 1: Apply the interface conditions. Every term of the form $F(t \mp x/c_j)$ has $u_{tt} = c_j^2 u_{xx}$. At the junction, displacement continuity gives $1 + R = D$. The slope condition gives

$$\frac{\mathcal{T}}{c_1}(R - 1)F'(t) = -\frac{\mathcal{T}}{c_2}DF'(t), \quad \boxed{Z_1(1 - R) = Z_2D.}$$

Since F is nonconstant, these amplitude equations follow.

Step 2: Solve and interpret the amplitudes. Solving the two equations gives

$$\boxed{R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad D = \frac{2Z_1}{Z_1 + Z_2}.}$$

Here $Z_j = \sqrt{\mathcal{T}\mu_j}$. A denser second string gives $R < 0$, an inverted reflected displacement. A lighter string gives $R > 0$. Equal impedances give no reflection and $D = 1$.

Step 3: Account for energy flux. The incident rightward power is $Z_1(F')^2$, the reflected leftward power has magnitude $Z_1R^2(F')^2$, and transmitted power is $Z_2D^2(F')^2$. The cross terms in $-\mathcal{T}u_tu_x$ on the incident side cancel. For a pulse with finite nonzero $\int(F')^2$, integration in time therefore gives

$$\boxed{\mathcal{R} = R^2, \quad \mathcal{D} = \frac{Z_2}{Z_1}D^2, \quad \mathcal{R} + \mathcal{D} = \frac{(Z_1 - Z_2)^2 + 4Z_1Z_2}{(Z_1 + Z_2)^2} = 1.}$$

Step 4: Test examples and extreme impedances. If $\mu_2 = 9\mu_1$, then $Z_2 = 3Z_1$, so $R = -1/2$, $D = 1/2$, $\mathcal{R} = 1/4$ and $\mathcal{D} = 3/4$. As $Z_2/Z_1 \rightarrow 0$, $R \rightarrow 1$ and $D \rightarrow 2$, but $\mathcal{D} \rightarrow 0$. As $Z_2/Z_1 \rightarrow \infty$, $R \rightarrow -1$, $D \rightarrow 0$ and again $\mathcal{D} \rightarrow 0$. These are the free-end and fixed-end reflection limits, respectively. Energy depends on impedance as well as squared displacement amplitude.