

Question 2

For the whole-line wave equation $u_{tt} = c^2 u_{xx}$, $c > 0$, you may use

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Take $f = 0$ and $g(s) = (1 - s^2)^2$ for $|s| < 1$, $g(s) = 0$ for $|s| \geq 1$, in consistent nondimensional units. Here g is continuously differentiable.

Tasks

1. Differentiate the integral formula to verify the equation and both initial conditions for these data.
2. Find the exact center displacement $u(0, t)$ for all $t \geq 0$.
3. Determine where the solution is zero and when a fixed point reaches the constant plateau behind the departing waves. Explain why compact initial velocity can leave nonzero displacement behind.
4. Sketch the profiles for $c = 1$ at $t = 1/2, 3/2, 3$. Check that the plateau height does not grow without bound and distinguish zero displacement from zero velocity.

Question 2 – Solution

Strategy. Initial velocity contributes an integral over a growing interval; it is not simply two displaced copies of the initial displacement.

Step 1: Verify the integral solution. Leibniz differentiation gives

$$u_t = \frac{g(x+ct) + g(x-ct)}{2}, \quad u_{tt} = \frac{c}{2}(g'(x+ct) - g'(x-ct)), \quad u_{xx} = \frac{1}{2c}(g'(x+ct) - g'(x-ct)).$$

Thus $u_{tt} = c^2 u_{xx}$. At $t = 0$, the integral is zero and $u_t(x, 0) = g(x)$. The C^1 regularity of g makes these second derivatives continuous.

Step 2: Integrate at the center. Symmetry gives $u(0, t) = c^{-1} \int_0^{\min(ct, 1)} (1-s)^2 ds$. Consequently

$$u(0, t) = \begin{cases} t - \frac{2c^2 t^3}{3} + \frac{c^4 t^5}{5}, & 0 \leq ct \leq 1, \\ \frac{8}{15c}, & ct \geq 1. \end{cases}$$

The two expressions agree at $ct = 1$.

Step 3: Locate the moving fronts and plateau. For $t > 0$, the integral is positive exactly when $[x-ct, x+ct]$ overlaps $(-1, 1)$ in positive length. It is therefore zero for $|x| \geq 1+ct$. When $ct \geq |x|+1$, the integration interval contains the entire support of g , so

$$u(x, t) = \frac{1}{2c} \int_{-1}^1 g(s) ds = \frac{8}{15c}.$$

The same value holds at equality because the endpoints of g contribute nothing. Behind the fronts, the string is displaced but locally stationary and flat. Displacement itself need not return to zero after a velocity pulse.

Step 4: Interpret the plotted profiles. Since $g \geq 0$, every interval integral lies between zero and the full integral. Thus $0 \leq u \leq 8/(15c)$ for every x, t . For $c = 1$, the fronts are at $x = \pm(1+t)$; a plateau appears once $t \geq 1$ and occupies $|x| \leq t-1$. Its height stays $8/15$ as its width grows. Zero local velocity on that plateau does not mean zero local displacement.

