

Question 1

An ideal string has constant tension $\mathcal{T} > 0$ and mass per unit length $\mu > 0$. Its small transverse displacement is $u(x, t)$. Assume $|u_x| \ll 1$, negligible longitudinal motion and bending stiffness, and a transverse applied force $p(x, t)$ per unit length. A linear drag force per unit length is $-bu_t$, with $b \geq 0$.

Tasks

1. Apply transverse momentum balance on an arbitrary interval and derive the linear wave model, explaining the small-slope approximation.
2. State the units of $\mu, \mathcal{T}, p, b$ and check each term. Identify the undamped wave speed and the effect of quadrupling μ at fixed tension.
3. Explain why both initial displacement and initial velocity are needed. Give two different undamped, source-free solutions on the whole line with the same zero initial displacement.
4. For $p = b = 0$, test $u = F(x - vt)$ with $F \in C^2$. Determine when this represents a traveling wave of the string and identify the exceptional profiles that do not determine v .

Question 1 – Solution

Strategy. Transverse acceleration is driven by the difference of vertical tension components, not by a first time derivative.

Step 1: Derive the momentum equation. The vertical tension component is $\mathcal{T} \sin \theta \approx \mathcal{T} u_x$ under the small-slope approximation. For an interval $[a, d]$,

$$\int_a^d \mu u_{tt} dx = \mathcal{T} u_x(d, t) - \mathcal{T} u_x(a, t) + \int_a^d (p - bu_t) dx.$$

Localization gives

$$\mu u_{tt} + bu_t = \mathcal{T} u_{xx} + p.$$

The linearization neglects changes of length and tension of higher order in the slope.

Step 2: Check dimensions and speed. The units are $\mu : \text{kg/m}$, $\mathcal{T} : \text{N}$, $p : \text{N/m}$ and $b : \text{Ns/m}^2$. Since u is a length, each term in the local equation has units N/m. Without drag or forcing,

$$u_{tt} = c^2 u_{xx}, \quad c = \sqrt{\mathcal{T}/\mu}.$$

The speed is in m/s. Quadrupling μ at fixed tension halves c .

Step 3: Explain the two initial data. Acceleration determines how velocity changes, so displacement alone cannot specify the motion. For example, on the whole line, both $u = 0$ and $u = Vt$, with constant $V \neq 0$, solve the source-free undamped equation and have $u(x, 0) = 0$. Their initial velocities differ. A prescribed pair $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ removes this particular ambiguity.

Step 4: Verify a moving profile. For $u = F(x - vt)$, $u_{tt} = v^2 F''$ and $u_{xx} = F''$. Hence

$$(v^2 - c^2)F'' = 0.$$

If F'' is not identically zero, the admissible speeds are $v = \pm c$, corresponding to opposite directions. If F is affine, both second derivatives vanish for every v ; a translating straight line cannot by itself identify a propagation speed. A non-affine smooth profile is needed to draw that inference from the PDE.