

Question 10

Divide an insulated homogeneous rod into three equal cells of width Δx . Let U_i^n be cell temperatures and $r = \alpha \Delta t / (\Delta x)^2 \geq 0$. Approximate each internal rightward flux by $-k(U_{i+1}^n - U_i^n) / \Delta x$, set outer fluxes to zero, and use a forward Euler time step for the cell energy balances.

Tasks

1. Derive the three-cell update matrix and prove that the sum of temperatures is conserved.
2. Find the exact range of r for which every update preserves the range of every input vector. Prove necessity as well as sufficiency.
3. Using the orthogonal vectors $(1, 1, 1)$, $(1, 0, -1)$ and $(1, -2, 1)$, determine exactly when $\sum_i (U_i^n - \bar{U})^2$ cannot increase in one step.
4. Apply one step with $r = 3/5$ to $(0, 1, 0)$. Check heat conservation and squared norm, and sketch the old and new cell values. Explain why energy stability need not imply preservation of nonnegative temperatures.

Question 10 – Solution

Strategy. Conservation, preservation of the temperature range and decay of a quadratic norm impose different conditions on a numerical scheme.

Step 1: Balance each cell. For the first cell, inflow minus outflow gives $U_1^{n+1} = U_1^n + r(U_2^n - U_1^n)$. Applying the same balance to the others yields

$$U^{n+1} = \begin{pmatrix} 1-r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-r \end{pmatrix} U^n.$$

Every column sums to one, so $\sum_i U_i$ is conserved. Equal cell volumes and constant capacity make this proportional to total heat. Every row also sums to one, so constant profiles are fixed.

Step 2: Characterize range preservation. All entries are nonnegative exactly when $0 \leq r \leq 1/2$. Each new value is then a convex combination of old values, preserving their range. For any $r > 1/2$, input $(0, 1, 0)$ gives a middle value $1 - 2r < 0$, outside $[0, 1]$. Thus the exact range is $\boxed{0 \leq r \leq 1/2}$.

Step 3: Test the quadratic energy on each mode. The stated mutually orthogonal vectors have update factors 1 , $1 - r$ and $1 - 3r$, respectively. The mean is fixed; the remaining squared mode components do not grow exactly when $|1 - r| \leq 1$ and $|1 - 3r| \leq 1$. For $r \geq 0$, this is

$$\boxed{0 \leq r \leq 2/3}.$$

Necessity follows by choosing the offending eigenvector as input. This threshold belongs to this particular three-cell insulated matrix.

Step 4: Exhibit the distinction numerically. At $r = 3/5$,

$$\boxed{(0, 1, 0) \mapsto (3/5, -1/5, 3/5)}.$$

Both sums equal one. The uncentered squared norm drops from 1 to $9/25 + 1/25 + 9/25 = 19/25$; since the mean is unchanged, the centered squared norm also drops. Nevertheless the middle temperature becomes negative. The scheme is quadratically stable here but fails the discrete range principle. The stem plot shows individual cell values, without suggesting a smooth interpolated temperature profile between cell centers.

