

Question 9

Two homogeneous rods have lengths L and $2L$, the same $k, C > 0$, insulated sides and no sources. At both ends each rod exchanges heat with the same ambient T_a by an outward convection law. The first rod has coefficient $h > 0$. The two initial profiles have the same shape as functions of relative position. Set $\alpha = k/C$ and choose a nonzero temperature scale ΔT .

Tasks

1. Nondimensionalize the first problem using $\xi = x/L$, $\eta = \alpha t/L^2$ and $\theta = (u - T_a)/\Delta T$. Identify the dimensionless boundary parameter.
2. Verify the PDE and initial shape for the proposed rescaling $u_2(x, t) = u_1(x/2, t/4)$ on the longer rod.
3. Derive exactly which convection coefficient h_2 makes this rescaling satisfy both boundary laws.
4. Assess the claim that doubling length always quadruples every thermal response time when the same h is used. State when the proposed scaling is valid, including the perfectly insulated case.

Question 9 – Solution

Strategy. Match the dimensionless boundary conditions as well as the interior diffusion time scale.

Step 1: Identify the dimensionless problem. The chain rule gives $\theta_\eta = \theta_{\xi\xi}$. The outward convection laws become

$$\theta_\xi(0, \eta) = \text{Bi} \theta(0, \eta), \quad -\theta_\xi(1, \eta) = \text{Bi} \theta(1, \eta), \quad \text{Bi} = \frac{hL}{k}.$$

This dimensionless coefficient is the Biot number for the chosen length scale. The interior time scale is L^2/α , but the boundary parameter also affects the dimensionless response.

Step 2: Verify the rescaled interior field. For $u_2(x, t) = u_1(x/2, t/4)$,

$$(u_2)_t = \frac{1}{4}(u_1)_t, \quad (u_2)_{xx} = \frac{1}{4}(u_1)_{xx}.$$

Thus the same diffusivity equation holds. At $t = 0$, relative positions agree: $x/(2L) = (x/2)/L$, so the stated initial-shape requirement is met.

Step 3: Match both convection conditions. At the left endpoint,

$$k(u_2)_x(0, t) = \frac{1}{2}k(u_1)_x(0, t/4) = \frac{h}{2}(u_2(0, t) - T_a).$$

The right outward derivative scales by the same factor. Therefore the coefficient making this a rescaling of the full problem is

$$h_2 = h/2, \quad h_2(2L)/k = hL/k.$$

Both rods then have the same dimensionless boundary problem.

Step 4: Qualify the length-squared claim. If $h_2 = h$, the longer rod instead has Biot number $2hL/k$. The proposed field generally fails the boundary law; the full response is not obtained merely by multiplying time by four. A special zero-excess equilibrium cannot establish a general scaling rule. The factor-four rescaling is valid when convection is also adjusted to $h/2$ and the initial shapes match. If both rods are perfectly insulated ($h = 0$), the zero-flux conditions are preserved automatically. Interior dimensional analysis alone does not justify ignoring changed boundary parameters.