

Question 8

Let u be continuous on $[0, L] \times [0, T]$, sufficiently differentiable in the interior, and satisfy $u_t = \alpha u_{xx}$ with $\alpha > 0$. Assume $0 \leq u(x, 0) \leq M$ and $0 \leq u(0, t), u(L, t) \leq M$, where $M > 0$.

Tasks

1. Prove $u \leq M$ by applying a maximum argument to $w = u - M - \varepsilon t$ and then letting $\varepsilon \downarrow 0$. Address a maximum occurring at the final observation time.
2. Prove $u \geq 0$ and explain why these bounds concern the global range rather than monotonic cooling at every fixed position.
3. Test $v(x, t) = Me^{\alpha(\pi/L)^2 t} \sin(\pi x/L)$ against the source-free heat equation and the range bound.
4. Find the source needed to realize this growing candidate in $Cv_t = kv_{xx} + s$, with $k/C = \alpha$. Give and verify a source-free decaying candidate with the same initial and endpoint data.

Question 8 – Solution

Strategy. A small time-dependent perturbation makes the maximum argument strict, while a growing test field exposes a missing source.

Step 1: Prove the upper bound. For $\varepsilon > 0$, w is nonpositive initially and at the spatial endpoints. If it has a positive maximum on the space-time rectangle, that maximum occurs at an interior spatial point and a time $t_* > 0$. There $w_{xx} \leq 0$ and $w_t \geq 0$: at the latest time, use the derivative from earlier times; at an interior time it is zero. Yet

$$w_t - \alpha w_{xx} = -\varepsilon < 0,$$

a contradiction. One can first apply the argument on any shorter time rectangle and pass to T by continuity if needed. Thus $u \leq M + \varepsilon t$, and letting $\varepsilon \downarrow 0$ gives $u \leq M$.

Step 2: Prove the lower bound. Apply the same argument to $-u$ with upper bound zero to obtain $-u \leq 0$. Hence $0 \leq u \leq M$ throughout the rectangle. The theorem does not say $u_t \leq 0$ at every point. At a local temperature minimum with positive curvature, the equation gives $u_t = \alpha u_{xx} > 0$. Redistribution can warm a cooler location without exceeding the allowed global range.

Step 3: Diagnose the growing sine. Write $\mu = (\pi/L)^2$. Then $v_t = \alpha \mu v$ and $v_{xx} = -\mu v$, so

$$v_t - \alpha v_{xx} = 2\alpha \mu v \neq 0$$

in the spatial interior. It satisfies zero endpoint values and the allowed initial profile, but for $t > 0$ its midpoint exceeds M . This violates the conclusion because it fails the source-free PDE hypothesis.

Step 4: Restore the missing physics or correct the time factor. The required source is $s = C(v_t - \alpha v_{xx}) = 2k\mu v > 0$ in the interior. With that source, the earlier source-free maximum argument no longer applies. Alternatively,

$$\tilde{v}(x, t) = M e^{-\alpha \mu t} \sin(\pi x/L)$$

has $\tilde{v}_t = -\alpha \mu \tilde{v} = \alpha \tilde{v}_{xx}$, the same initial profile and zero endpoints. It remains between zero and M .