

Question 6

On an infinite homogeneous rod, consider

$$u(x, t) = T_a + \frac{B}{\sqrt{t + \tau}} \exp\left(-\frac{x^2}{4\alpha(t + \tau)}\right), \quad B, \tau, \alpha > 0, \quad t \geq 0.$$

Use $C > 0$ for volumetric heat capacity and interpret integrated heat per unit cross-sectional area. You may use $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$ and $\int_{\mathbb{R}} z^2 e^{-z^2} dz = \sqrt{\pi}/2$.

Tasks

1. Verify $u_t = \alpha u_{xx}$ directly, retaining the derivative of the time-dependent prefactor.
2. Compute the total excess heat per area and show that it is constant.
3. Compute the variance of the excess-temperature profile about zero and describe its peak and width as time increases.
4. Determine where the temperature is increasing at a given time, and the time of maximum temperature at a fixed point x . Sketch the profiles for $\alpha = B = \tau = 1$ at $t = 0, 3$, marking the inflection points.

Question 6 – Solution

Strategy. Differentiate the amplitude and width together; local warming can coexist with a falling global peak and conserved heat.

Step 1: Verify the differential equation. Put $v = u - T_a$ and $\sigma = t + \tau > 0$. Logarithmic differentiation gives

$$v_t = v \left(-\frac{1}{2\sigma} + \frac{x^2}{4\alpha\sigma^2} \right), \quad v_{xx} = v \left(-\frac{1}{2\alpha\sigma} + \frac{x^2}{4\alpha^2\sigma^2} \right).$$

Thus $v_t = \alpha v_{xx}$. Omitting the derivative of $\sigma^{-1/2}$ would remove the first term and invalidate the equation.

Step 2: Integrate the excess heat. With $z = x/\sqrt{4\alpha\sigma}$,

$$\mathcal{H} = C \int_{\mathbb{R}} v \, dx = 2CB\sqrt{\pi\alpha}.$$

The factor from the increasing width cancels the decreasing prefactor. The Gaussian flux tends to zero at both infinities, consistent with no net heat loss from the infinite rod.

Step 3: Measure spreading. Symmetry gives center zero. The provided Gaussian integrals yield

$$\frac{\int_{\mathbb{R}} x^2 v \, dx}{\int_{\mathbb{R}} v \, dx} = 2\alpha\sigma, \quad \max_x v = \frac{B}{\sqrt{\sigma}}.$$

Thus the root-mean-square width grows as $\sqrt{2\alpha(t + \tau)}$, while the peak decreases as $(t + \tau)^{-1/2}$. The inflection points occur at $x = \pm\sqrt{2\alpha\sigma}$.

Step 4: Locate local warming. Since $v > 0$, $u_t > 0$ precisely when $x^2 > 2\alpha(t + \tau)$. A fixed point reaches its maximum on $t \geq 0$ at

$$t_* = \max\left(0, \frac{x^2}{2\alpha} - \tau\right).$$

Outside the initial inflection points it first warms and then cools. Inside them it cools from the start; equality gives an initially zero derivative. Therefore diffusion does not force temperature to decrease at every location. The graph uses excess temperature v , so the ambient offset is removed.

