

Question 4

Two homogeneous layers in series have thicknesses L_1, L_2 , conductivities $k_1, k_2 > 0$ and the same cross-sectional area. The outer temperatures are $T_h > T_c$. There are no sources. A contact resistance per area $R \geq 0$ satisfies $u_- - u_+ = Rq$, where u_-, u_+ are the left and right interface traces and q is the steady rightward heat flux. The interface stores no energy.

Tasks

1. Explain why flux is continuous at the interface while temperature may jump. State all equations determining the steady profiles.
2. Derive q, u_-, u_+ and the two linear temperature profiles.
3. In consistent nondimensional units, take $L_1 = L_2 = 1$, $k_1 = 1$, $k_2 = 2$, $R = 1/2$, $T_h = 100$ and $T_c = 0$. Compute and sketch the profiles, showing both interface traces.
4. Analyze the limits $R \rightarrow 0$ and $R \rightarrow \infty$. Explain why imposing temperature continuity when $R > 0$ would generally contradict the model.

Question 4 – Solution

Strategy. Add the two layer resistances and the contact resistance, retaining the interface temperature drop.

Step 1: Apply the interface balance. No interfacial storage or source means incoming flux equals outgoing flux. Thus the same q satisfies $u_x = -q/k_1$ in the first layer and $u_x = -q/k_2$ in the second. The boundary and contact relations are

$$u(0) = T_h, \quad u(L_1 + L_2) = T_c, \quad u_- - u_+ = Rq.$$

Temperature continuity is a special case of zero resistance, not the general interface law.

Step 2: Add all temperature drops. Writing $\mathcal{R} = L_1/k_1 + R + L_2/k_2$ gives

$$q = \frac{T_h - T_c}{\mathcal{R}}, \quad u_- = T_h - \frac{qL_1}{k_1}, \quad u_+ = T_c + \frac{qL_2}{k_2}.$$

The profiles are $T_h - qx/k_1$ on the first layer and $T_c + q(L_1 + L_2 - x)/k_2$ on the second. Their traces differ by Rq .

Step 3: Verify the numerical example. Here $\mathcal{R} = 1 + 1/2 + 1/2 = 2$, so

$$q = 50, \quad u_- = 50, \quad u_+ = 25, \quad u(x) = \begin{cases} 100 - 50x, & 0 \leq x < 1, \\ 50 - 25x, & 1 < x \leq 2. \end{cases}$$

The flux is 50 on both sides, although the slopes differ. The temperature jump is $25 = Rq$. Open circles show separate traces, not a single assigned temperature at an idealized zero-thickness contact.

Step 4: Take the resistance limits. As $R \rightarrow 0$, the flux tends to $(T_h - T_c)/(L_1/k_1 + L_2/k_2)$ and the traces meet. As $R \rightarrow \infty$, $q \rightarrow 0$, each layer becomes nearly constant at its own outer temperature, and $Rq \rightarrow T_h - T_c$. A large resistance can therefore sustain a finite temperature jump while the flux vanishes. For finite $R > 0$, forcing $u_- = u_+$ would give $q = 0$, contradicting $T_h > T_c$ and the finite total resistance.

