

Question 3

A rod has constant area, insulated sides, no internal source and conductivity $k(x) = k_0(1 + x/L)$, with $k_0, L > 0$. Its end temperatures are $u(0) = T_h$, $u(L) = T_c$, with $T_h > T_c$. The steady equation is $(k(x)u_x)_x = 0$, and flux per area toward the right is $q = -k(x)u_x$.

Tasks

1. Derive the exact temperature and heat flux.
2. Verify the equation and both boundary values, and locate the largest magnitude of the temperature gradient.
3. Test the straight-line interpolant between the same endpoints in the correct variable-conductivity equation. Explain its physical failure.
4. Define k_{eff} by $q = k_{\text{eff}}(T_h - T_c)/L$. Compute it and compare it with the arithmetic spatial average of k . Sketch the normalized exact and linear temperature profiles.

Question 3 – Solution

Strategy. Constant steady heat flux requires the gradient to vary inversely with conductivity.

Step 1: Integrate the flux law. Since $(ku_x)_x = 0$, the flux q is constant. Integrating $u_x = -q/k(x)$ gives

$$T_c - T_h = -\frac{qL}{k_0} \log 2.$$

Consequently

$$q = \frac{k_0(T_h - T_c)}{L \log 2}, \quad u(x) = T_h - (T_h - T_c) \frac{\log(1 + x/L)}{\log 2}.$$

The flux is positive: heat travels from the hotter left end toward the right.

Step 2: Check the profile and gradient. The logarithm is zero at 0 and $\log 2$ at L , giving the two endpoint values. Also

$$u_x = -\frac{T_h - T_c}{L \log 2 (1 + x/L)}, \quad k(x)u_x = -\frac{k_0(T_h - T_c)}{L \log 2}.$$

Thus the product is constant as required. The gradient magnitude is largest at $x = 0$ and decreases by a factor of two from left to right.

Step 3: Diagnose the linear candidate. For $\ell = T_h - (T_h - T_c)x/L$, one has $\ell_{xx} = 0$, but

$$(k\ell_x)_x = -\frac{k_0(T_h - T_c)}{L^2} \neq 0.$$

Its rightward flux grows along the rod, so more heat leaves each small interval than enters. Without a source this profile would cool locally, not remain steady. Using $ku_{xx} = 0$ discards the essential term $k'u_x$.

Step 4: Interpret the effective conductivity. The definition gives $k_{\text{eff}} = k_0/\log 2$, while the arithmetic average is $3k_0/2$. In fact $\log 2 > 2/3$: the function $1/t$ is strictly convex and its average on $[1, 2]$ exceeds its midpoint value $2/3$. Hence

$$k_{\text{eff}} = \frac{k_0}{\log 2} < \frac{3k_0}{2}.$$

Steady conduction uses the reciprocal average $k_{\text{eff}} = L/\int_0^L k^{-1} dx$. The graph uses $\xi = x/L$ and $\theta = (u - T_c)/(T_h - T_c)$.

