

Question 1

A stationary rod has cross-sectional area $A(x) > 0$, conductivity $k(x) > 0$ and volumetric heat capacity $C(x) > 0$. Temperature $u(x, t)$ is uniform across each section; the lateral surface is insulated. A volumetric source $s(x, t)$ has units W/m^3 . Define signed heat power toward increasing x by $Q = -kAu_x$, in watts. All coefficients are independent of time.

Tasks

1. Apply energy conservation to an arbitrary interval $[a, b]$ and derive the local heat equation. State the units of every term in the balance.
2. Expand the spatial derivative and identify the diffusivity. Explain when the equation reduces to $u_t = \alpha u_{xx}$.
3. After nondimensionalization, take $0 < x < 1$, $A = e^x$, $k = C = 1$, $s = 0$ and an instantaneous profile $u = x$. Find Q , the storage rate Au_t and u_t . Sketch the first two quantities.
4. Explain why this linear temperature profile need not be steady, even though its curvature is zero. Distinguish heat flux per area from total heat power.

Question 1 – Solution

Strategy. Balance total heat power before dividing by the local cross-sectional area.

Step 1: Derive the conservation law. The stored thermal energy relative to a fixed reference has time derivative $\int_a^b CAu_t dx$. Inflow minus outflow plus generation gives

$$\int_a^b CAu_t dx = Q(a, t) - Q(b, t) + \int_a^b As dx.$$

Since this holds on every interval, with Fourier's law it yields

$$CAu_t = -Q_x + As = (kAu_x)_x + As.$$

Each local term has units W/m: C is J/(m³K), A is m² and u_t is K/s. Each integrated term, including $Q(a) - Q(b)$, is in watts.

Step 2: Identify the geometric contribution. Dividing by CA gives

$$u_t = \frac{k}{C}u_{xx} + \frac{(kA)'}{CA}u_x + \frac{s}{C}.$$

The diffusivity k/C has units m²/s. For a homogeneous constant-area rod without a source, this is $u_t = \alpha u_{xx}$ with constant $\alpha = k/C$. More generally, the first-derivative term vanishes when kA is constant; constant diffusivity additionally requires k/C constant.

Step 3: Evaluate the nondimensional example. Here $u_x = 1$ and $u_{xx} = 0$, but

$$Q(x) = -e^x, \quad Au_t = -Q_x = e^x, \quad u_t = 1.$$

The signed power is negative because heat moves toward decreasing temperature, which here is the negative x direction. Its magnitude increases toward the larger cross section.

Step 4: Interpret the nonzero storage. Flux per unit area is $q = -ku_x = -1$, constant in this example. Total power is $Q = Aq = -e^x$, which is not constant. More heat enters a small interval through its larger right face than leaves through its smaller left face, so energy accumulates. Zero curvature alone is a steady-state test only under the coefficient and geometry assumptions that remove the extra term.

