

Question 10

Let $f \in L^2(-\pi, \pi)$ be real and periodic with mean m and Fourier coefficients a_n, b_n . For $t > 0$, define the damped expansion

$$U_t(x) = m + \sum_{n=1}^{\infty} e^{-n^2 t} (a_n \cos nx + b_n \sin nx).$$

Use Parseval and L^2 convergence of Fourier partial sums. This problem concerns the series itself; no partial differential equation is assumed.

Tasks

1. Prove that for each fixed $t > 0$ the series, and every series obtained by differentiating in x finitely many times, converge uniformly.
2. Prove $U_t \rightarrow f$ in L^2 as $t \downarrow 0$ using a finite-head and small-tail argument, with an exact coefficient formula for the error.
3. If $\sum n^2(a_n^2 + b_n^2) \leq M$, prove $\|U_t - f\|_2 \leq \sqrt{\pi M t}$.
4. Show that the mean is preserved and that $\|U_t - m\|_2 \leq \|f - m\|_2$. Explain why the general L^2 assumption alone cannot guarantee uniform convergence to the assigned function f as $t \downarrow 0$.

Question 10 – Solution

Strategy. Damping controls all high frequencies for positive t ; removing it requires a separate convergence argument.

Step 1: Prove smoothness for positive damping. Parseval gives $E = \sum (a_n^2 + b_n^2) < \infty$. After k derivatives, the absolute value of the n th harmonic is at most $n^k e^{-n^2 t} \sqrt{a_n^2 + b_n^2}$. Cauchy-Schwarz bounds the sum by

$$\sqrt{E} \left(\sum_{n \geq 1} n^{2k} e^{-2n^2 t} \right)^{1/2} < \infty.$$

The exponential dominates every power, so the Weierstrass test applies for each fixed $t > 0$ and each integer $k \geq 0$. Repeated use of the uniform derivative theorem justifies differentiation; U_t is smooth and periodic.

Step 2: Remove damping in the integral norm. The Fourier coefficients of the difference give

$$\|U_t - f\|_2^2 = \pi \sum_{n \geq 1} (1 - e^{-n^2 t})^2 (a_n^2 + b_n^2).$$

Given $\varepsilon > 0$, choose K so that $\pi \sum_{n > K} (a_n^2 + b_n^2) < \varepsilon/2$. Since $0 \leq 1 - e^{-n^2 t} \leq 1$, this bounds the error tail for every t . The remaining finite sum tends to zero, hence is below $\varepsilon/2$ for sufficiently small t . This proves the limit.

Step 3: Obtain a quantitative rate under extra information. For $z \geq 0$, $1 - e^{-z} \leq \min(z, 1)$, so $(1 - e^{-z})^2 \leq z$. Applying this with $z = n^2 t$ gives

$$\|U_t - f\|_2^2 \leq \pi t \sum n^2 (a_n^2 + b_n^2) \leq \pi M t.$$

Taking square roots yields the requested rate; this uses a stronger hypothesis than mere square summability.

Step 4: Separate smoothing from uniform recovery. Uniform integration of the damped series preserves the mean m . Parseval gives $\|U_t - m\|_2^2 = \pi \sum e^{-2n^2 t} (a_n^2 + b_n^2) \leq \|f - m\|_2^2$. Every U_t is continuous. Choose an L^2 periodic step function with a jump: it cannot be a uniform limit of these continuous functions. Even isolated redefinitions of f are invisible to the coefficients. Thus smooth approximants and mean-square recovery do not alone guarantee uniform recovery of an arbitrarily assigned representative.