

Question 8

Let real coefficient sequences satisfy

$$\sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) \leq M, \quad M > 0.$$

Define the formal series $f = m + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$ and let S_N be its degree- N truncation, $N \geq 1$. Write $T_N = \sum_{n > N} n^{-2}$.

Tasks

1. Prove that the series defines a continuous periodic function by establishing uniform convergence.
2. Prove $\|f - S_N\|_{\infty} \leq \sqrt{MT_N} < \sqrt{M/N}$.
3. For a fixed N and prescribed point x_0 , construct coefficients satisfying the constraint for which the bound $\sqrt{MT_N}$ is attained.
4. Prove $\|f - S_N\|_2^2 \leq \pi M / (N+1)^2$ and construct an equality case for this bound. Explain why the two optimizing coefficient patterns differ.

Question 8 – Solution

Strategy. Use the weighted coefficient budget with two different dual estimates: one for a point value and one for energy.

Step 1: Prove uniform convergence from the budget. For any finite tail from p to q , Cauchy–Schwarz gives

$$\left| \sum_{n=p}^q (a_n \cos nx + b_n \sin nx) \right| \leq \left(\sum_{n=p}^q n^2 (a_n^2 + b_n^2) \right)^{1/2} \left(\sum_{n=p}^q \frac{\cos^2 nx + \sin^2 nx}{n^2} \right)^{1/2}.$$

It is at most $\sqrt{M}(\sum_{n \geq p} n^{-2})^{1/2}$, independently of x , and tends to zero as $p \rightarrow \infty$. The uniform Cauchy criterion applies. The limit of these continuous periodic partial sums is continuous and periodic.

Step 2: Bound the infinite tail. Passing to the uniform limit of the finite-tail inequality gives

$$\|f - S_N\|_\infty \leq \sqrt{MT_N} < \sqrt{M/N} \quad (N \geq 1).$$

The weighted hypothesis controls the full collection of omitted modes, not just a finite list of observed coefficients.

Step 3: Attain the exact pointwise bound. Set every coefficient with $n \leq N$ to zero and, for $n > N$, choose

$$a_n = \sqrt{\frac{M}{T_N}} \frac{\cos(nx_0)}{n^2}, \quad b_n = \sqrt{\frac{M}{T_N}} \frac{\sin(nx_0)}{n^2}.$$

The weighted sum equals M , and at x_0 the tail equals $\sqrt{MT_N}$. This proves sharpness of the first bound over all admissible sequences. The mean m may be arbitrary because it cancels from the error.

Step 4: Optimize the integral error. Orthogonality, justified by the uniform convergence, gives

$$\|f - S_N\|_2^2 = \pi \sum_{n > N} (a_n^2 + b_n^2) \leq \frac{\pi}{(N+1)^2} \sum_{n > N} n^2 (a_n^2 + b_n^2) \leq \boxed{\frac{\pi M}{(N+1)^2}}.$$

Equality is attained by $a_{N+1} = \sqrt{M}/(N+1)$ and all other coefficients zero. The integral optimizer concentrates its budget in the cheapest omitted mode; the pointwise optimizer distributes it across all omitted modes with phases aligned at the chosen point.