

Question 7

For real bounded piecewise continuous 2π -periodic f , its degree-one Fourier projection is

$$(S_1 f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)(1+2\cos t) dt.$$

Use the essential supremum norm, which ignores changes on sets of measure zero.

Tasks

1. Prove $\|S_1 f\|_{\infty} \leq \Lambda \|f\|_{\infty}$, where $\Lambda = (2\pi)^{-1} \int_{-\pi}^{\pi} |1+2\cos t| dt$.
2. Compute Λ exactly, locating every sign change of the kernel.
3. Construct an admissible f with $\|f\|_{\infty} = 1$ for which equality is attained at $x = 0$. Verify its mean and first cosine coefficient directly.
4. Use the result to give a sharp worst-case amplification factor for a bounded input error under S_1 . Explain why orthogonal projection being a contraction in L^2 does not imply contraction in the supremum norm.

Question 7 – Solution

Strategy. The signed Fourier kernel controls integral projection, while its absolute integral controls worst-case pointwise error.

Step 1: Bound the projection in the supremum norm. Taking absolute values in the given integral gives

$$|S_1 f(x)| \leq \frac{\|f\|_\infty}{2\pi} \int_{-\pi}^{\pi} |1 + 2 \cos t| dt.$$

Null-set changes do not affect this estimate or the projected polynomial. Taking the supremum over x proves the claimed inequality.

Step 2: Integrate the absolute kernel. The kernel is nonnegative for $|t| \leq 2\pi/3$ and negative outside. Its full signed integral is 2π . Its integral on the two negative intervals is

$$2 \int_{2\pi/3}^{\pi} (1 + 2 \cos t) dt = \frac{2\pi}{3} - 2\sqrt{3}.$$

Changing the sign of that contribution gives

$$\Lambda = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} > 1.$$

Step 3: Attain the bound with an explicit input. Choose $f(t) = \operatorname{sgn}(1 + 2 \cos t)$, with value zero at its two zeros. It is even and piecewise constant with essential supremum one. Its positive region has length $4\pi/3$, so its mean is $1/3$. Direct integration gives

$$a_1 = \frac{2}{\pi} \left(\int_0^{2\pi/3} \cos t dt - \int_{2\pi/3}^{\pi} \cos t dt \right) = \frac{2\sqrt{3}}{\pi}, \quad b_1 = 0.$$

Hence $S_1 f(0) = 1/3 + 2\sqrt{3}/\pi = \Lambda$, proving sharpness.

Step 4: Interpret the stability statement. For input error e with $\|e\|_\infty \leq \varepsilon$, linearity gives $\|S_1(f + e) - S_1 f\|_\infty \leq \Lambda \varepsilon$. The choice $e = \varepsilon \operatorname{sgn}(1 + 2 \cos t)$ attains this bound. Orthogonal projection still satisfies $\|S_1 e\|_2 \leq \|e\|_2$ by the Pythagorean identity. That statement uses an integral norm, and does not prevent a larger pointwise peak. Contraction is a property of a specified norm.