

Question 6

For $N \geq 1$, consider real trigonometric polynomials $p(x) = A_0 + \sum_{n=1}^N (A_n \cos nx + B_n \sin nx)$ on $[-\pi, \pi]$. Use $\|p\|_2^2 = \int_{-\pi}^{\pi} p^2$ and define $D_N(x) = 1 + 2 \sum_{n=1}^N \cos nx$.

Tasks

1. Prove $\|p\|_{\infty} \leq \sqrt{(2N+1)/(2\pi)} \|p\|_2$ by an orthonormal-basis argument.
2. Prove the constant is sharp and characterize all nonzero equality cases at a prescribed point x_0 .
3. For $p_N = D_N/(2N+1)$, compute both norms exactly and determine its pointwise limit. Explain what this says about point evaluation under mean-square convergence.
4. Derive the sine-quotient form of D_N and sketch p_2, p_8 over a period. Explain why this sequence is not the sequence of Fourier partial sums of one fixed function.

Question 6 – Solution

Strategy. Point evaluation on a finite-dimensional space has a norm that grows with the number of available modes.

Step 1: Apply Cauchy–Schwarz in coefficient space. An orthonormal basis is $1/\sqrt{2\pi}$, $\cos nx/\sqrt{\pi}$, $\sin nx/\sqrt{\pi}$. The sum of their squared values at any point is

$$\frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^N (\cos^2 nx + \sin^2 nx) = \frac{2N+1}{2\pi}.$$

Cauchy–Schwarz against the orthonormal coefficient vector proves

$$\|p\|_{\infty} \leq \sqrt{\frac{2N+1}{2\pi}} \|p\|_2.$$

Step 2: Identify equality cases. Equality at x_0 holds exactly when the coefficient vector is a real nonzero multiple of the basis-value vector at x_0 . Equivalently,

$$p(x) = CD_N(x - x_0), \quad C \in \mathbb{R} \setminus \{0\}.$$

This exhibits the sharp constant, not merely an upper estimate.

Step 3: Build a vanishing-energy sequence with a fixed peak. The triangle inequality and the value at zero give $\|D_N\|_{\infty} = 2N+1$. Orthogonality gives $\|D_N\|_2^2 = 2\pi(2N+1)$. Consequently

$$\|p_N\|_{\infty} = 1, \quad \|p_N\|_2^2 = \frac{2\pi}{2N+1} \rightarrow 0.$$

At 0 modulo 2π , $p_N = 1$. At every other fixed point it tends to zero by the quotient below. Point evaluation is therefore not continuous in the L^2 norm on the union of these polynomial spaces.

Step 4: Explain the oscillatory peak. Summing $\sum_{n=-N}^N e^{inx}$ yields

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)} \quad (x \notin 2\pi\mathbb{Z}).$$

The removable value at zero is $2N+1$. The normalized peak narrows with N . These p_N cannot be successive partial sums of one fixed function: their constant coefficients $1/(2N+1)$ change with N , whereas truncation preserves every already-included Fourier coefficient.

