

Question 4

Let $G(x) = (\pi - x)/2$ on $0 < x < 2\pi$, extended periodically with value zero at the joins. Its Fourier coefficients give

$$G(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi).$$

This identity follows from the piecewise smooth Fourier theorem. Set $P_N = \sum_{n=1}^N \sin(nx)/n$ and fix $0 < \delta < \pi$.

Tasks

1. Use a finite geometric sum to bound $\left| \sum_{n=p}^q \sin nx \right|$ uniformly in p, q for $x \in [\delta, 2\pi - \delta]$.
2. Apply summation by parts to prove $\|G - P_N\|_{\infty, [\delta, 2\pi - \delta]} \leq 1/((N+1)\sin(\delta/2))$.
3. Prove that convergence is not uniform even on the full open interval $(0, 2\pi)$. Explain why the preceding bound cannot be used with $\delta = 0$.
4. Compute the squared $L^2(0, 2\pi)$ error and bound it. Sketch the error for $N = 8$, marking the endpoint limits separately from the assigned endpoint errors.

Question 4 – Solution

Strategy. Cancellation controls a tail away from a jump, but the cancellation bound degenerates near the join.

Step 1: Bound finite oscillatory blocks. For $x \notin 2\pi\mathbb{Z}$,

$$\left| \sum_{n=p}^q e^{inx} \right| = \left| \frac{e^{ipx}(1 - e^{i(q-p+1)x})}{1 - e^{ix}} \right| \leq \frac{1}{|\sin(x/2)|}.$$

The same bound holds for the imaginary part. On the specified closed interval, it is at most $B = 1/\sin(\delta/2)$.

Step 2: Sum by parts with a decreasing weight. Put $A_k = \sum_{n=N+1}^k \sin nx$, so $|A_k| \leq B$. Then

$$\sum_{n=N+1}^M \frac{\sin nx}{n} = \frac{A_M}{M} + \sum_{k=N+1}^{M-1} A_k \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

The absolute value is bounded by $B/(N+1)$. Passing to the convergent tail gives

$$\sup_{\delta \leq x \leq 2\pi - \delta} |G - P_N| \leq \frac{1}{(N+1)\sin(\delta/2)} \rightarrow 0.$$

Step 3: Locate the global obstruction. For every fixed N , $P_N(x) \rightarrow 0$ as $x \downarrow 0$, while $G(x) \rightarrow \pi/2$. Therefore $\sup_{0 < x < 2\pi} |G - P_N| \geq \pi/2$ for every N . Excluding the endpoint itself does not exclude points arbitrarily close to it. At $\delta = 0$ the geometric-sum bound has a zero denominator and gives no finite uniform control.

Step 4: Verify convergence in the integral norm. The sine functions have squared norm π over this period. Hence

$$\|G - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^2} < \frac{\pi}{N}.$$

The error tends to zero in mean square despite persistent near-jump suprema. Its assigned endpoint errors are zero, whereas its interior limits are $\pi/2$ at 0 and $-\pi/2$ at 2π .

