

Question 3

You may use the Fourier identity

$$F(x) = \frac{\pi^2}{6} - \frac{\pi|x|}{2} + \frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad -\pi \leq x \leq \pi,$$

extended periodically. Let $P_N = \sum_{n=1}^N \cos(nx)/n^2$, $N \geq 1$.

Tasks

1. Prove uniform convergence and find the exact value of $\|F - P_N\|_{\infty}$ as a positive numerical series.
2. Determine all points where this maximum absolute error occurs, modulo 2π .
3. Prove strict two-sided bounds using integrals and determine the smallest integer N for which the uniform error is less than $1/100$.
4. Find the exact squared mean-square error as a series and compare its decay rate with that of the uniform error. Sketch $F - P_5$ and its upper and lower uniform-error levels.

Question 3 – Solution

Strategy. Positive coefficients make the usual triangle bound attain its value at a common phase.

Step 1: Turn the tail estimate into an equality. The Weierstrass test applies because $\sum n^{-2} < \infty$. Put $T_N = \sum_{n>N} n^{-2}$. For every x , $|F(x) - P_N(x)| \leq T_N$, while at $x = 0$ every cosine equals one. Hence

$$\|F - P_N\|_{\infty} = T_N.$$

This is an exact error, not merely a sufficient upper bound.

Step 2: Identify every equality point. For the tail to equal T_N , every positive summand must have $\cos nx = 1$. Using two consecutive integers $n > N$ forces $x = 0$ modulo 2π . For the tail to equal $-T_N$, every $\cos nx$ would have to be -1 . But then $\cos(2nx) = 1$ for a frequency also in the tail, a contradiction. Thus the maximum absolute error occurs exactly at $x = 0$ modulo 2π .

Step 3: Certify the minimal truncation. Strict monotonicity of t^{-2} gives

$$\frac{1}{N+1} < T_N < \frac{1}{N}.$$

At $N = 100$, $T_N < 1/100$. At $N = 99$, $T_N > 1/100$; monotonicity excludes every smaller N . Therefore the exact smallest choice is $N = 100$. The two integral bounds straddle the threshold tightly enough to decide it.

Step 4: Compare the two norms. Orthogonality gives

$$\|F - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^4}, \quad \frac{\pi}{3(N+1)^3} < \|F - P_N\|_2^2 < \frac{\pi}{3N^3}.$$

The uniform error is of order N^{-1} , whereas the L^2 error is of order $N^{-3/2}$ (its square is of order N^{-3}). Distinguish the norm from its square.

