

Question 2

For integers $m \geq 1$, let

$$h_m(x) = \max(1 - m|x|, 0), \quad -\pi \leq x \leq \pi,$$

and extend periodically. Write $S_N^{(m)}$ for the degree- N Fourier partial sum of this particular function. The target changes with m ; these are not partial sums of one fixed function.

Tasks

1. Compute $\|h_m\|_\infty$, $\|h_m\|_2^2$ and the pointwise limit as $m \rightarrow \infty$. Distinguish mean-square convergence to zero from pointwise convergence to zero.
2. Derive the mean and all Fourier coefficients of h_m .
3. For each fixed m , prove $S_N^{(m)} \rightarrow h_m$ uniformly as $N \rightarrow \infty$. State a bound whose dependence on m is explicit.
4. Evaluate both iterated limits at $x = 0$, first letting N tend to infinity and then reversing the order. Explain why they differ, and sketch h_1, h_3, h_9 near the origin.

Question 2 – Solution

Strategy. A shrinking support can make integral errors small while retaining a fixed peak.

Step 1: Compare the modes of convergence. The peak is always one and direct integration gives

$$\|h_m\|_\infty = 1, \quad \|h_m\|_2^2 = 2 \int_0^{1/m} (1 - mx)^2 dx = \frac{2}{3m}.$$

The pointwise limit is one at 0 modulo 2π and zero elsewhere. Thus $h_m \rightarrow 0$ in mean square and almost everywhere, but not at every point and not uniformly. The pointwise limit has a single-point spike in each period.

Step 2: Compute the spectrum. The function is even, so $b_n = 0$. Its area is $1/m$, giving mean $1/(2\pi m)$. Integration over its support gives

$$a_n = \frac{2}{\pi} \int_0^{1/m} (1 - mx) \cos nx dx = \frac{2m}{\pi n^2} (1 - \cos(n/m)).$$

For each fixed n , $0 \leq a_n \leq 1/(\pi m)$ by $1 - \cos u \leq u^2/2$. Therefore every fixed coefficient tends to zero as $m \rightarrow \infty$.

Step 3: Keep the target fixed when taking a Fourier limit. For fixed m , $|a_n| \leq 4m/(\pi n^2)$, so the series converges absolutely and uniformly. The piecewise smooth Fourier theorem identifies its sum with the continuous target h_m . In particular,

$$\|h_m - S_N^{(m)}\|_\infty \leq \frac{4m}{\pi N}.$$

The factor m prevents using this bound uniformly over all targets.

Step 4: Compare the iterated limits. For fixed m , the inner N -limit at zero is $h_m(0) = 1$. For fixed N , the mean and finitely many coefficients all tend to zero. Thus

$$\lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty} S_N^{(m)}(0) = 1, \quad \lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} S_N^{(m)}(0) = 0.$$

Fixed-mode convergence does not control the collective contribution of modes whose number grows with m . There is no common uniform estimate permitting this interchange.

