

**Question 1**

Define a  $2\pi$ -periodic function by  $f = -1$  on  $(-\pi, 0)$ ,  $f = 2$  on  $(0, \pi)$ ,  $f(0) = 7$  and  $f(\pi) = -4$ . Let  $S_N$  be its real Fourier partial sum with mean  $a_0/2$ . Use the piecewise smooth Fourier theorem (the sum is the average of the one-sided limits) and Parseval. Here  $\|u\|_2^2 = \int_{-\pi}^{\pi} |u|^2$ .

**Tasks**

1. Derive the mean and all coefficients. Determine the pointwise series sum everywhere, including both types of assigned exceptional values.
2. Compute  $\|f - S_N\|_2^2$  as a coefficient tail and prove it tends to zero.
3. Prove that  $S_N$  does not converge uniformly to  $f$ . Also prove a lower bound of  $3/2$  for the essential supremum error, which ignores sets of measure zero.
4. Decide whether redefining only the jump values can restore uniform convergence. Sketch  $f$  and  $S_9$ , displaying the assigned values separately from the open one-sided limits.

## Question 1 – Solution

**Strategy.** Distinguish the assigned function, its Fourier pointwise limit and the equivalence class measured by an integral norm.

**Step 1: Compute coefficients and limits.** Apart from isolated values,  $f = 1/2 + (3/2) \operatorname{sgn}(\sin x)$ . Thus  $a_0 = 1$ , every positive cosine coefficient is zero, and

$$b_n = \frac{3(1 - (-1)^n)}{\pi n} = \begin{cases} 6/(\pi n), & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

The series sum equals  $-1$  or  $2$  on the open intervals and  $1/2$  at every jump. It does not equal either assigned value  $7$  or  $-4$  there.

**Step 2: Verify mean-square convergence.** Isolated values do not affect integrals. Parseval and orthogonality give

$$\|f - S_N\|_2^2 = \frac{36}{\pi} \sum_{\substack{n > N \\ n \text{ odd}}} \frac{1}{n^2} \leq \frac{36}{\pi N} \rightarrow 0.$$

The coefficient tail is square summable despite failure at the assigned jump values.

**Step 3: Separate point and essential errors.** Every  $S_N(0) = 1/2$ , so  $\|f - S_N\|_\infty \geq 13/2$  for every  $N$ . Moreover, by continuity of  $S_N$ , as  $x \downarrow 0$  its error against the right-hand value  $2$  tends to  $3/2$ . For any  $\varepsilon > 0$ , that error exceeds  $3/2 - \varepsilon$  on an interval of positive length. Hence

$$\operatorname{ess\,sup} |f - S_N| \geq 3/2.$$

Removing isolated exceptional values cannot remove this obstruction.

**Step 4: Test possible redefinitions.** The one-sided limits remain  $-1$  and  $2$  after any change only at the jumps. The resulting target is still discontinuous. A uniform limit of the continuous trigonometric polynomials  $S_N$  would be continuous, so no such redefinition restores uniform convergence. Mean-square convergence and the open-interval pointwise limits are unchanged. Filled points in the graph show the assigned values.

