

Question 10

Two periodic signals are $f(x) = \cos x + \cos 2x$ and $g(x) = \sin x + \cos 2x$. To align them by one delay, minimize

$$D(\delta) = \int_0^{2\pi} [f(x - \delta) - g(x)]^2 dx, \quad \delta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Tasks

1. Use orthogonality to derive $D(\delta)$ without expanding a long pointwise square.
2. Determine all globally optimal delays and the exact minimum error. Prove global optimality.
3. Compare delays 0 and $\pi/2$. Explain why exactly aligning the first harmonic need not give the best alignment of the full signal.
4. Decide whether any delay makes the two functions identical. Sketch $D(\delta)/\pi$ over one period, marking every minimizer, and explain the nonuniqueness of the best delay.

Question 10 – Solution

Strategy. One delay rotates each harmonic by a different multiple of the same angle, creating a coupled optimization.

Step 1: Compute distance through correlation. Both signals have squared norm 2π . Angle addition and orthogonality give

$$\int_0^{2\pi} f(x-\delta)g(x) dx = \pi(\sin \delta + \cos 2\delta).$$

Therefore

$$\frac{D(\delta)}{\pi} = 4 - 2(\sin \delta + \cos 2\delta).$$

The sine sign follows from $\cos(x-\delta) = \cos x \cos \delta + \sin x \sin \delta$.

Step 2: Optimize over the whole circle. Put $t = \sin \delta \in [-1, 1]$. Since $\cos 2\delta = 1 - 2t^2$,

$$\frac{D(\delta)}{\pi} = \frac{7}{4} + 4 \left(t - \frac{1}{4} \right)^2.$$

The feasible value $t = 1/4$ is the unique minimizing sine value. Consequently

$$D_{\min} = \frac{7\pi}{4}, \quad \delta = \arcsin(1/4) \text{ or } \pi - \arcsin(1/4) \pmod{2\pi}.$$

Completing the square proves these are all global minimizers.

Step 3: Compare two tempting alignments. At $\delta = 0$, the second harmonics agree and $D = 2\pi$. At $\delta = \pi/2$, the first harmonic matches $\sin x$, but the second becomes $-\cos 2x$, so $D = 4\pi$. Exactly matching one harmonic can worsen the full error. The optimum balances their contributions and improves on both choices.

Step 4: Interpret the two minima. The minimum is strictly positive, so no delay makes the signals identical. The objective depends only on $\sin \delta$; the two distinct delays with sine $1/4$ give the same compromise. They do not produce the same shifted function: their first cosine coefficients are opposite and nonzero. Thus nonuniqueness of the best delay does not imply exact agreement or identical aligned waveforms. The graph shows both minima in $[0, 2\pi)$.

