

Question 9

For real a, b , a modulated waveform is

$$s_{a,b}(x) = (1 + a \cos x) \cos 3x + b \sin x \sin 3x.$$

Use a 2π Fourier expansion, and define average power as $(2\pi)^{-1} \int_0^{2\pi} s_{a,b}^2$.

Tasks

1. Derive the complete Fourier spectrum using product identities. Identify the frequencies surrounding the carrier frequency 3.
2. Recover a, b from measured cosine coefficients C_2, C_4 . Determine all choices canceling the frequency-4 term while leaving a prescribed frequency-2 coefficient d .
3. For $d = 1/2$ with frequency 4 canceled, find the fundamental period. Prove minimality even though the frequency-1 coefficient is zero.
4. Compute the average power for general a, b , then for this special case. Prove a sharp global bound on the special waveform's absolute value and identify points where equality holds.

Question 9 – Solution

Strategy. Resolve the products into actual harmonics before inferring frequency content, period or energy.

Step 1: Resolve the two products. The identities $\cos x \cos 3x = (\cos 2x + \cos 4x)/2$ and $\sin x \sin 3x = (\cos 2x - \cos 4x)/2$ give

$$s_{a,b}(x) = \cos 3x + \frac{a+b}{2} \cos 2x + \frac{a-b}{2} \cos 4x.$$

The mean and every sine coefficient are zero. The neighboring frequencies, often called sidebands, are 2 and 4, not frequency 1 from the modulation alone.

Step 2: Invert the coefficient measurements. The linear equations give

$$a = C_2 + C_4, \quad b = C_2 - C_4.$$

Requiring $C_4 = 0$ and $C_2 = d$ fixes $a = b = d$ uniquely. The two modulation terms reinforce the lower frequency and cancel the upper frequency when their amplitudes agree.

Step 3: Establish the least period. For $d = 1/2$ the signal is $s = \cos 3x + \frac{1}{2} \cos 2x$. If $T > 0$ is a period, translating this finite expansion and using orthogonality forces $3T$ and $2T$ each to be integer multiples of 2π . Subtracting gives T an integer multiple of 2π . Conversely 2π works. Thus the fundamental period is $\boxed{2\pi}$. A nonzero first harmonic is sufficient for this period in a 2π polynomial, but is not necessary: frequencies 2 and 3 already enforce it.

Step 4: Measure power and peak amplitude. Orthogonality gives

$$P = \frac{1}{2} \left(1 + \frac{(a+b)^2}{4} + \frac{(a-b)^2}{4} \right) = \frac{1}{2} + \frac{a^2 + b^2}{4}.$$

At $a = b = 1/2$, $P = 5/8$. Also $|s| \leq |\cos 3x| + \frac{1}{2} |\cos 2x| \leq 3/2$, with equality at $x = 2k\pi$. For negative equality both cosines would have to be -1 ; the equations $3x = (2j+1)\pi$, $2x = (2k+1)\pi$ are incompatible by parity. Positive equality requires both to be 1, giving exactly $x = 2k\pi$.