

Question 8

Let $f(x) = 1 + \cos x + 2 \sin x + \cos 2x$. Its periodic autocorrelation is

$$R(\tau) = \frac{1}{2\pi} \int_0^{2\pi} f(x)f(x+\tau) dx.$$

Tasks

1. Derive R using orthogonality. Explain why sine terms disappear from the correlation even though f has a nonzero sine coefficient.
2. Find the exact maximum and minimum of R , including all shifts attaining them modulo 2π .
3. Construct a real trigonometric polynomial g with the same mean and the same R which is not a translate of f . Prove both claims.
4. Derive the identity relating R to the squared distance between f and its translate. Determine every shift leaving f unchanged and connect this with the fundamental period.

Question 8 – Solution

Strategy. Correlation retains harmonic energies but discards their individual phases.

Step 1: Evaluate the correlation. Different frequencies are orthogonal. At frequency n , a pair $a_n \cos nx + b_n \sin nx$ contributes $(a_n^2 + b_n^2) \cos(n\tau)/2$; the two mixed sine terms cancel. The constant contributes the square of the mean. Thus

$$R(\tau) = 1 + \frac{5}{2} \cos \tau + \frac{1}{2} \cos 2\tau.$$

In particular, $R(0) = 4$ is the mean square of f , not the square of its mean. The correlation is even regardless of the parity of f .

Step 2: Optimize over all shifts. Put $t = \cos \tau \in [-1, 1]$. Then $R = t^2 + (5/2)t + 1/2$, whose derivative $2t + 5/2$ is positive throughout this interval. Hence

$$\max R = 4 \quad (\tau = 0 \bmod 2\pi), \quad \min R = -1 \quad (\tau = \pi \bmod 2\pi).$$

A correlation can be negative here; the original signal is not required to be nonnegative.

Step 3: Exhibit phase ambiguity beyond translation. Take $g(x) = 1 + \sqrt{5} \cos x + \cos 2x$. Its mean is one and its harmonic squared amplitudes are 5 and 1, so it gives exactly the same R . If $g(x) = f(x + c)$, equality at frequency two requires $\cos 2c = 1$ and $\sin 2c = 0$, so $c = k\pi$. But then the first sine coefficient of $f(x + c)$ is $2(-1)^k \neq 0$, unlike g . Thus g is not a translate. Correlation does not determine relative harmonic phases.

Step 4: Recover the exact translation invariance. Translation preserves $\int f^2$. Expanding the squared difference gives

$$\frac{1}{2\pi} \int_0^{2\pi} [f(x + \tau) - f(x)]^2 dx = 2(R(0) - R(\tau)).$$

This is zero precisely at $\tau = 0$ modulo 2π . Zero integral of a continuous nonnegative square means equality at every point. Therefore the invariant shifts are exactly $2k\pi$, and the fundamental period is 2π .