

Question 7

For real parameters α, β , seek a twice continuously differentiable 2π -periodic solution of

$$y'' + 4y = 3 + \cos x + \alpha \cos 2x + \beta \sin 2x.$$

Tasks

1. Derive necessary conditions on α, β by multiplying by each resonant mode and integrating over one period.
2. Prove those conditions are sufficient and find all periodic solutions when they hold.
3. Under the same conditions, impose $y(0) = y'(0) = 0$. Find the resulting solution and verify all conditions directly.
4. Explain why initial conditions cannot restore periodic solvability when a resonant forcing coefficient is nonzero. Relate this to the failed division by $4 - n^2$ in the Fourier equations.

Question 7 – Solution

Strategy. Test the forcing against the periodic homogeneous modes before dividing by a spectral factor.

Step 1: Derive the compatibility conditions. For a periodic C^2 function, both y and y' match at the period endpoints. Two integrations by parts therefore give

$$\int_0^{2\pi} (y'' + 4y) \cos 2x = 0, \quad \int_0^{2\pi} (y'' + 4y) \sin 2x = 0.$$

Orthogonality makes the corresponding right sides $\pi\alpha$ and $\pi\beta$. Thus

$$\alpha = \beta = 0$$

is necessary. The constant and first harmonic do not contribute to these tests.

Step 2: Construct every compatible solution. With these parameters zero, a particular solution is $3/4 + \cos x/3$. The homogeneous equation has basis $\cos 2x, \sin 2x$, both periodic. Thus

$$y(x) = \frac{3}{4} + \frac{1}{3} \cos x + A \cos 2x + B \sin 2x, \quad A, B \in \mathbb{R}.$$

This proves sufficiency as well as completeness: subtract the particular solution from any solution and solve the homogeneous equation.

Step 3: Enforce the additional normalization. The initial values give $3/4 + 1/3 + A = 0$ and $2B = 0$. Hence

$$y(x) = \frac{3}{4} + \frac{1}{3} \cos x - \frac{13}{12} \cos 2x.$$

It has $y(0) = 0$, $y'(0) = 0$, and is 2π -periodic together with all derivatives. Substitution gives $y'' + 4y = 3 + \cos x$ because the frequency-two term is annihilated.

Step 4: Identify the resonance obstruction. For frequency n , the mode equation multiplies each coefficient of y by $4 - n^2$. At $n = 2$ it reads $0 = \alpha$ or $0 = \beta$: it is a compatibility condition, not a formula for an unknown coefficient. When it holds, the two coefficients are free until extra conditions are supplied. When it fails, no periodic C^2 solution exists. Although an initial-value problem still has a unique solution, its resonant response cannot be periodic; altering homogeneous constants does not change either integral obstruction.