

Question 5

A change of origin replaces $f(x)$ by $f(t + c)$, where c is defined modulo 2π . Consider

$$f(x) = 2 \cos x + 2 \sin x + \cos 2x.$$

Tasks

1. Derive how a general coefficient pair (a_n, b_n) changes under $x = t + c$. Prove that its amplitude is unchanged.
2. Determine every origin c for which $f(t + c)$ is even, or prove that none exists.
3. Determine every origin c for which $f(t + c)$ is odd. Give the shifted expansion for one such origin.
4. Replace $\cos 2x$ by $\sin 2x$ to obtain g . Find all origins making $g(t + c)$ even and give one resulting cosine-only expansion. Explain why separately aligning each harmonic is not a valid test for a single common symmetry center.

Question 5 – Solution

Strategy. A common origin must satisfy the phase conditions of every nonzero harmonic simultaneously.

Step 1: Derive the coefficient rotation. Angle addition gives the new coefficients

$$a'_n = a_n \cos nc + b_n \sin nc, \quad b'_n = -a_n \sin nc + b_n \cos nc.$$

Squaring and adding cancels the cross terms, so $(a'_n)^2 + (b'_n)^2 = a_n^2 + b_n^2$. For a trigonometric polynomial, evenness is equivalent to every b'_n being zero; oddness requires a zero mean and every a'_n zero. Necessity follows from orthogonality, and sufficiency follows from sine/cosine parity.

Step 2: Test all evenness conditions. At frequency one, $b'_1 = 2(\cos c - \sin c) = 0$ implies $c = \pi/4 + k\pi$. At frequency two, $b'_2 = -\sin 2c$ would then equal -1 . Hence

$$\text{No origin makes } f(t+c) \text{ even.}$$

The first frequency can be aligned, but that shift makes the second purely sine.

Step 3: Find every oddness origin. The first condition $a'_1 = 2(\cos c + \sin c) = 0$ gives $c = 3\pi/4 + k\pi$. At each of these, $a'_2 = \cos 2c = 0$, and the mean is already zero. These are all solutions. At $c = 3\pi/4$,

$$f(t + 3\pi/4) = -2\sqrt{2} \sin t + \sin 2t.$$

Modulo 2π the two origins are $3\pi/4$ and $7\pi/4$.

Step 4: Change the relative harmonic phase. For $g = 2 \cos x + 2 \sin x + \sin 2x$, the first-frequency evenness condition is again $c = \pi/4 + k\pi$. Now the second is $b'_2 = \cos 2c = 0$, which is satisfied. At $c = \pi/4$,

$$g(t + \pi/4) = 2\sqrt{2} \cos t + \cos 2t.$$

All evenness origins are $\pi/4 + k\pi$. Each harmonic admits its own phase alignment, but a symmetry of the full function requires the same c for all frequencies. The two examples have identical harmonic amplitudes and different answers to the evenness question.