

Question 4

Let $f(x) = e^x$ for $0 < x < 2$ and extend periodically with period 2. At each join assign the average $(1+e^2)/2$. Write

$$f \sim m + \sum_{n \geq 1} (a_n \cos(n\pi x) + b_n \sin(n\pi x)), \quad m = \frac{1}{2} \int_0^2 f, \quad (a_n, b_n) = \int_0^2 f(\cos(n\pi x), \sin(n\pi x)).$$

Set $E = e^2 - 1$.

Tasks

1. Derive m, a_n, b_n . Identify the series sum at the periodic join.
2. Use integration by parts on $f' = f$ inside $(0, 2)$ to obtain two coefficient relations. Explain why omitting the boundary jump would incorrectly force every positive-frequency coefficient to zero.
3. Find a continuous 2-periodic function F with $F' = f - m$ away from the joins and $F(0) = 0$. Prove uniqueness under this normalization.
4. Compute the mean and all Fourier coefficients of F directly. Verify that differentiating its positive-frequency terms formally gives those of $f - m$, and explain why f itself cannot have a continuous periodic primitive.

Question 4 – Solution

Strategy. Keep the endpoint jump in integration by parts, and remove the mean before seeking a periodic primitive.

Step 1: Compute the full coefficients. For $k = n\pi$, exponential-trigonometric antiderivatives at $0, 2$ give

$$m = \frac{E}{2}, \quad a_n = \frac{E}{1+k^2}, \quad b_n = -\frac{kE}{1+k^2}.$$

The Fourier sum at a join is $(1+e^2)/2$, since its two limits are e^2 and 1 . The assigned join value does not enter any integral.

Step 2: Retain the boundary contribution. On the open interval, $f' = f$. Integration by parts gives

$$\int_0^2 f' \cos kx = E + k \int_0^2 f \sin kx, \quad \int_0^2 f' \sin kx = -k \int_0^2 f \cos kx.$$

Thus $a_n = E + kb_n$ and $b_n = -ka_n$, yielding the coefficients above. Dropping E would give $(1+k^2)a_n = 0$ and then $b_n = 0$. The periodic extension has a jump, so it is not a globally smooth periodic solution of $f' = f$.

Step 3: Construct the periodic primitive. Integrating the mean-zero forcing gives, on $[0, 2]$,

$$F(x) = e^x - 1 - \frac{E}{2}x.$$

Both endpoint values are zero, so its periodic extension is continuous. Two such primitives differ by a constant on each open period; continuity connects those constants, and $F(0) = 0$ fixes the remaining freedom.

Step 4: Check the transformed coefficients. Direct integration gives mean $M = -1$. If A_n, B_n denote the coefficients of F , the polynomial terms have zero cosine integral and $\int_0^2 x \sin kx = -2/k$. Consequently

$$A_n = \frac{E}{1+k^2}, \quad B_n = \frac{E}{k(1+k^2)}.$$

Indeed $kB_n = a_n$ and $-kA_n = b_n$. These are coefficient identities; the derivative equation is verified by the closed form on each open period. Finally, a continuous piecewise smooth periodic primitive of f would have net change $\int_0^2 f = E \neq 0$, which is impossible. Subtracting the mean is essential.