

Question 3

Consider a half-wave rectifier with output $h(x) = \max(\sin x, 0)$, extended with period 2π . Use the mean-term convention $a_0/2$. You may use Parseval:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n \geq 1} (a_n^2 + b_n^2).$$

Tasks

1. Express h using $\sin x$ and $|\sin x|$. Derive its complete real Fourier series.
2. Explain why all sine coefficients except b_1 vanish, and why only even positive cosine frequencies occur.
3. Compute the average squared output and its division into the constant term, the fundamental harmonic, and all higher harmonics.
4. Determine the fundamental periods of h and $|\sin x|$, with a proof that the proposed periods are minimal. Sketch h and its degree-8 Fourier sum.

Question 3 – Solution

Strategy. Rectification combines an odd fundamental with an even function of half the period.

Step 1: Derive the expansion. Since $h = (\sin x + |\sin x|)/2$, its mean is $1/\pi$. Direct integration on $(0, \pi)$ gives $b_1 = 1/2$ and $b_n = 0$ for $n \neq 1$. For $k \geq 1$, product-to-sum integration gives

$$\frac{1}{\pi} \int_0^\pi \sin x \cos(2kx) dx = -\frac{2}{\pi(4k^2 - 1)}.$$

Odd cosine coefficients vanish, including a_1 . Thus

$$h(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2 - 1}.$$

The coefficients are absolutely summable; the standard piecewise smooth Fourier theorem identifies the sum with this continuous target, including its corners.

Step 2: Explain the selection rules. The function $|\sin x|$ is even, so contributes only cosines. It is also π -periodic, eliminating odd cosine frequencies in a 2π expansion. The remaining odd part is exactly $\sin x/2$, so it contributes only b_1 . The full rectified waveform is neither even nor odd.

Step 3: Account for the average power. Direct integration gives $(2\pi)^{-1} \int h^2 = 1/4$. Orthogonality separates the nonnegative contributions:

$$P_{\text{constant}} = \frac{1}{\pi^2}, \quad P_{\text{fundamental}} = \frac{1}{8}, \quad P_{\text{higher}} = \frac{1}{8} - \frac{1}{\pi^2}.$$

They sum to $1/4$. The higher-harmonic contribution is positive; the mean squared is only the constant contribution, not the total average squared output.

Step 4: Prove the fundamental periods. The maxima of h occur exactly at $\pi/2 + 2k\pi$. Any positive period must take one maximum to another, hence be a positive multiple of 2π ; 2π works. The maxima of $|\sin x|$ occur at $\pi/2 + k\pi$, so its least positive period is π . Thus the full output has fundamental period 2π despite its even-frequency cosine terms: its nonzero sine fundamental matters.

