

Question 2

A 2π -periodic pulse has height one on the circular interval $c - \pi/3 < x < c + \pi/3$ (modulo 2π), height zero elsewhere, and value $1/2$ at its two edges. The unknown center c is defined modulo 2π . Use the real Fourier convention $f = a_0/2 + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$, with coefficients integrated over any full period. The series has the usual one-sided-average value at a jump.

Tasks

1. Derive the mean and all coefficients by centering the integration interval at c .
2. Determine exactly which entire harmonics vanish. Distinguish a missing harmonic from a zero cosine coefficient caused only by phase.
3. Recover c modulo 2π from the signed pair (a_1, b_1) . State an exact consistency condition for a measured pair under this pulse model.
4. Show what is lost if only the harmonic amplitudes $\sqrt{a_n^2 + b_n^2}$ are measured. For $c = \pi/4$, sketch the pulse and its degree-12 partial sum, marking the edge values.

Question 2 – Solution

Strategy. Centering exposes the pulse width; translating back rotates each sine/cosine coefficient pair.

Step 1: Integrate the centered pulse. The area is $2\pi/3$, so the mean is $1/3$ and $a_0 = 2/3$. Put $x = c + t$. The sine terms odd in t integrate to zero, leaving

$$a_n = \frac{2 \sin(n\pi/3)}{\pi n} \cos nc, \quad b_n = \frac{2 \sin(n\pi/3)}{\pi n} \sin nc.$$

These formulas also apply when the pulse crosses the chosen period boundary: integrate instead over a period centered at c .

Step 2: Identify genuine missing modes. The amplitude is

$$R_n = \sqrt{a_n^2 + b_n^2} = \frac{2|\sin(n\pi/3)|}{\pi n}.$$

Hence the entire harmonic vanishes exactly when 3 divides n . For other n , a_n alone can vanish because $\cos nc = 0$, while $b_n \neq 0$. Discarding such a sine term would remove a real harmonic.

Step 3: Recover the circular center. For $n = 1$ the prefactor is positive:

$$(a_1, b_1) = \frac{\sqrt{3}}{\pi}(\cos c, \sin c), \quad c = \text{atan2}(b_1, a_1) \pmod{2\pi}.$$

Consistency is exactly $a_1^2 + b_1^2 = 3/\pi^2$ for real measured coefficients. Every pair on that circle corresponds to precisely one center modulo 2π . Using only an ordinary arctangent of b_1/a_1 loses quadrant information.

Step 4: Separate location from amplitude. Every R_n is independent of c , so amplitude measurements leave every center possible, even if all harmonics are known. At $c = \pi/4$, the pulse edges in $[-\pi, \pi]$ are $-\pi/12$ and $7\pi/12$. The sum is one between them, zero outside, and $1/2$ at each edge. The plotted partial sum uses both coefficient families; the finite oscillations do not alter these limiting edge averages.

