

Question 9

For $\alpha > 0$, consider

$$-u''_{\alpha} + \alpha^2 u_{\alpha} = 1 + \cos x, \quad u'_{\alpha}(0) = u'_{\alpha}(\pi) = 0.$$

The zeroth cosine mode has eigenvalue zero for $-d^2/dx^2$.

Tasks

1. Solve by cosine modes, treating the constant separately. Verify the equation and both boundary conditions.
2. Prove uniqueness for $\alpha > 0$ by an energy identity. Determine the mean and its behavior as $\alpha \downarrow 0$.
3. Subtract the mean to define $v_{\alpha} = u_{\alpha} - \pi^{-1} \int_0^{\pi} u_{\alpha}$. Find its limit and prove uniform convergence of the functions and their first two derivatives.
4. Explain why the unregularized problem with forcing $1 + \cos x$ has no solution, whereas the limit of v_{α} solves a different normalized problem. State that problem precisely.

Question 9 – Solution

Strategy. Track the constant mode separately: it becomes singular as the regularization vanishes.

Step 1: Solve the forced modes. The constant is divided by α^2 , while the first cosine is divided by $1 + \alpha^2$. Therefore

$$u_\alpha = \frac{1}{\alpha^2} + \frac{\cos x}{1 + \alpha^2}.$$

Its derivative is $-\sin x/(1 + \alpha^2)$, zero at both endpoints. Substitution recovers both the constant forcing one and the cosine term.

Step 2: Establish uniqueness and inspect the mean. For a homogeneous difference, integration by parts gives $\int_0^\pi [(w')^2 + \alpha^2 w^2] dx = 0$, so $w = 0$. The mean is $\boxed{1/\alpha^2}$ and diverges as $\alpha \downarrow 0$; the full family cannot have a finite uniform limit.

Step 3: Control the normalized family. Subtracting the mean gives $v_\alpha = \cos x/(1 + \alpha^2)$. The first two derivatives of cosine, as well as cosine itself, have supremum norm one. Thus for $j = 0, 1, 2$,

$$\|v_\alpha^{(j)} - (\cos x)^{(j)}\|_\infty = \frac{\alpha^2}{1 + \alpha^2} \rightarrow 0.$$

This proves function and derivative convergence on the whole interval.

Step 4: Identify the correct limiting problem. The original forcing has integral $\pi \neq 0$, violating the Neumann condition $\int f = 0$ when $\alpha = 0$. Hence the unregularized original problem is impossible. Subtracting the mean also removes the constant forcing: $-v_\alpha'' + \alpha^2 v_\alpha = \cos x$. The limit solves

$$\boxed{-v'' = \cos x, \quad v'(0) = v'(\pi) = 0, \quad \int_0^\pi v = 0, \quad v = \cos x.}$$

It is the unique zero-mean solution of this compatible problem.