

Question 7

For $0 < \rho < 1$, define the positive profile

$$P_\rho(x) = \frac{1 - \rho^2}{1 - 2\rho \cos x + \rho^2}, \quad 0 \leq x \leq \pi.$$

Use $P_\rho = a_0/2 + \sum_{n \geq 1} a_n \cos nx$ and, when needed, Parseval: $\int_0^\pi f^2 = \pi(a_0/2)^2 + (\pi/2) \sum_{n \geq 1} a_n^2$.

Tasks

1. Derive the cosine expansion from a geometric series, justifying uniform convergence for fixed ρ . Identify the mean and every coefficient.
2. Compute the integral and squared norm exactly. Locate the maximum and minimum and give their values.
3. Determine the pointwise limit as $\rho \uparrow 1$. Explain why integrating it does not recover the constant mass.
4. For every continuous φ on $[0, \pi]$, prove $\int_0^\pi P_\rho \varphi \rightarrow \pi \varphi(0)$ by splitting at a small fixed $\delta > 0$. Sketch $P_{1/2}$ and $P_{4/5}$ and interpret this limit.

Question 7 – Solution

Strategy. Sum the geometric coefficients exactly, then locate the mass accumulating near the endpoint.

Step 1: Derive the uniformly convergent expansion. The geometric identity gives $1 + 2 \operatorname{Re}[\rho e^{ix}/(1 - \rho e^{ix})] = P_\rho(x)$. Since $\sum \rho^n < \infty$, its cosine series converges absolutely and uniformly:

$$P_\rho = 1 + 2 \sum_{n=1}^{\infty} \rho^n \cos nx, \quad a_0 = 2, \quad a_n = 2\rho^n.$$

Step 2: Compute mass, energy and extrema. Uniform integration gives $\int_0^\pi P_\rho = \pi$. Parseval and the geometric sum give

$$\int_0^\pi P_\rho^2 = \pi \frac{1 + \rho^2}{1 - \rho^2}.$$

The denominator increases with x . The maximum at zero is $(1+\rho)/(1-\rho)$; the minimum at π is $(1-\rho)/(1+\rho)$.

Step 3: Explain the apparent loss of mass. For fixed $x > 0$, the numerator tends to zero and the denominator to $2(1 - \cos x) > 0$, so $P_\rho(x) \rightarrow 0$. At zero it tends to infinity. The almost-everywhere limit is zero, yet the integral stays π . Uniform convergence and interchange of the integral with this limit are not available; the exact squared norm also diverges.

Step 4: Prove endpoint concentration. Given $\eta > 0$, choose $\delta > 0$ so $|\varphi(x) - \varphi(0)| < \eta$ on $[0, \delta]$. That part of $\int P_\rho(\varphi - \varphi(0))$ is at most $\pi\eta$ in magnitude. On $[\delta, \pi]$, $P_\rho \leq (1 - \rho^2)/[2\rho(1 - \cos \delta)] \rightarrow 0$ uniformly, so the remaining part tends to zero. Letting $\eta \downarrow 0$ proves the claim. The limit is mass π concentrated at zero, not an ordinary integrable profile.

