

Question 6

Let $f = 1$ on $[0, \pi/2)$, $f(\pi/2) = 1/2$, and $f = 0$ on $(\pi/2, \pi]$. Write $C_N = a_0/2 + \sum_{n=1}^N a_n \cos nx$, including $C_0 = a_0/2$, and define

$$\sigma_N = \frac{1}{N+1} \sum_{j=0}^N C_j.$$

Let F be the even 2π -periodic extension. The ordinary Fourier convergence theorem may be used for this piecewise constant function.

Tasks

1. Compute the mean and coefficients. Show that C_1 becomes negative although the target is nonnegative.
2. Derive the coefficient weights in σ_N . Expand $K_N(t) = (N+1)^{-1} |\sum_{j=0}^N e^{ijt}|^2$ and prove nonnegativity and integral 2π over one period.
3. Verify $\sigma_N(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} F(x-t) K_N(t) dt$, then prove $0 \leq \sigma_N \leq 1$ everywhere.
4. Determine all limits of σ_N on $[0, \pi]$ by averaging convergent sequences of partial sums. Sketch C_9, σ_9 and the target, and explain the tradeoff made by averaging.

Question 6 – Solution

Strategy. Use an average with a nonnegative kernel and unit total mass to preserve the range of the data.

Step 1: Exhibit a negative partial sum. The mean is $1/2$ and $a_n = 2 \sin(n\pi/2)/(\pi n)$. Thus $C_1 = 1/2 + (2/\pi) \cos x$ has minimum $1/2 - 2/\pi < 0$ at π , despite $0 \leq f \leq 1$.

Step 2: Compute the weights and kernel. Mode n appears in $N + 1 - n$ of the partial sums, giving

$$\sigma_N = \frac{1}{2} + \sum_{n=1}^N \left(1 - \frac{n}{N+1}\right) a_n \cos nx.$$

Counting pairs with index difference n in the squared modulus gives $K_N = 1 + 2 \sum_{n=1}^N (1 - n/(N+1)) \cos nt$. The squared-modulus definition proves $K_N \geq 0$, and integrating the finite expansion gives $\int_{-\pi}^{\pi} K_N = 2\pi$.

Step 3: Prove preservation of the range. Insert the finite kernel expansion into the convolution. The average integral of $F(x-t) \cos nt$ is $a_n \cos nx/2$; after shifting variables, its sine component vanishes by evenness of F . The constant term gives mean $1/2$. This proves the formula. Because $0 \leq F \leq 1$ and the kernel is nonnegative with mass 2π ,

$$0 \leq \sigma_N(x) \leq 1.$$

Step 4: Identify the limits and tradeoff. Ordinary partial sums tend to one left of the jump, zero right of it, and $1/2$ at the jump. The extension is continuous at zero and π , with values one and zero. Arithmetic means of a convergent sequence have the same limit, so σ_N has exactly these limits. Averaging damps the higher modes and prevents overshoot, but spreads the finite transition instead of fitting the edge as sharply.

