

Question 4

For $f = e^x$ on $[0, 1]$, improve a cosine approximation by matching slopes:

$$\ell = x + \frac{e-1}{2}x^2, \quad r = e^x - \ell, \quad A_N = \ell + m_r + \sum_{n=1}^N c_n \cos(n\pi x),$$

where $m_r = \int_0^1 r$ and $c_n = 2 \int_0^1 r(x) \cos(n\pi x) dx$.

Tasks

1. Compute the raw mean and cosine coefficients of e^x . Find the endpoint slopes of every raw cosine partial sum.
2. Verify the slopes of ℓ , then compute m_r and all c_n . Explain the improvement from order n^{-2} to order n^{-4} .
3. Prove $\|e^x - A_N\|_\infty \leq 2(1+e)/(3\pi^4 N^3)$ and give an integer N certified to make the bound less than 10^{-4} .
4. Justify one differentiation of the residual series and verify the slopes of every A_N . Sketch the derivative errors of raw and corrected approximants for $N = 8$. State what slope matching does not guarantee.

Question 4 – Solution

Strategy. Remove the endpoint-slope contribution while retaining the residual's generally nonzero mean.

Step 1: Compute the raw data. For $k = n\pi$, direct integration gives

$$\text{mean} = e - 1, \quad a_n = \frac{2(e(-1)^n - 1)}{1 + k^2}.$$

Every raw finite cosine sum has zero endpoint derivatives, disagreeing with $f'(0) = 1$ and $f'(1) = e$.

Step 2: Cancel the slope contribution. The derivative $\ell' = 1 + (e - 1)x$ matches both target endpoint slopes. Two integrations by parts give its positive-frequency coefficient $2(e(-1)^n - 1)/k^2$; the constant second derivative has zero cosine integral. Subtracting yields

$$m_r = \frac{5}{6}(e - 1) - \frac{1}{2}, \quad c_n = -\frac{2(e(-1)^n - 1)}{k^2(1 + k^2)}.$$

The residual has zero endpoint derivatives, so the leading slope term cancels and fourth-power decay remains. Its mean must still be included.

Step 3: Certify the function error. The bound $|c_n| \leq 2(1 + e)/(\pi^4 n^4)$ gives uniform absolute convergence. The continuous piecewise smooth even extension identifies the sum with r . Consequently

$$\|e^x - A_N\|_\infty \leq \frac{2(1 + e)}{\pi^4} \int_N^\infty t^{-4} dt = \frac{2(1 + e)}{3\pi^4 N^3}.$$

The choice $N = 7$ makes this bound less than 10^{-4} .

Step 4: Verify the derivative behavior. Derivative coefficients are bounded by a constant times n^{-3} , so that series converges uniformly. Together with convergence at one point, this justifies differentiation. Every residual partial sum has zero endpoint slopes; adding ℓ gives $A'_N(0) = 1, A'_N(1) = e$ exactly. Slope matching does not force exact endpoint values at finite N ; those are controlled by the function-error bound and become exact in the limit.

